On the Kähler-Ricci flow with small initial E_1 energy (I)

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1 Introduction and main results

1.1 The motivation

In [9] [10], a family of functionals $E_k(k = 1, 2, \cdots n)$ was introduced by the first named author and G. Tian to prove the convergence of the Kähler-Ricci flow under appropriate curvature assumptions. The aim of this program (cf. [6]) is to study how the lower bound of E_1 is used to derive the convergence of the Kähler-Ricci flow, i.e., the existence of Kähler-Einstein metrics. We will address this question in Subsection 1.2. The corresponding problem of the relation between the lower bound of E_0, which is the K-energy introduced by T. Mabuchi, and the existence of Kähler-Einstein metrics has been extensively studied (cf. [2], [9], [10], [13]). One interesting question in this program is how the lower bound of E_1 compares to the lower bound of E_0. We will give a satisfactory answer to this question in Subsection 1.3.

1.2 The lower bound of E_1 and Kähler-Einstein metrics

Let $(M, [\omega])$ be a polarized compact Kähler manifold with $[\omega] = 2\pi c_1(M) > 0$ (the first Chern class) in this paper. In [6], the first named author proved a stability theorem of the Kähler-Ricci flow near the infimum of E_1 under the assumption that the initial metric has $Ric > -1$ and $|Rm|$ bounded. Unfortunately, this stability theorem needs a topological assumption

$$(-1)^n ([c_1(M)]^{[2]}[\omega]^{[n-2]} - \frac{2(n+1)}{n}[c_2(M)][\omega]^{[n-2]}) \geq 0. \tag{1.1}$$

The only known compact manifold which satisfies this condition is $\mathbb{C}P^n$, which restricts potential applications of this result. The main purpose of this paper is to remove this assumption.

Theorem 1.1. *Suppose that M is pre-stable, and E_1 is bounded from below in $[\omega]$. For any $\delta, \Lambda > 0$, there exists a small positive constant $\epsilon(\delta, \Lambda) > 0$ such that for any metric ω_0 in the subspace $\mathcal{A}(\delta, \Lambda, \omega, \epsilon)$ of Kähler metrics*

$$\{\omega_\phi = \omega + \sqrt{-1}\partial\bar\partial\phi \mid Ric(\omega_\phi) > -1 + \delta, \ |Rm|(\omega_\phi) \leq \Lambda, \ E_1(\omega_\phi) \leq \inf E_1 + \epsilon\},$$

where $E_1(\omega') = E_{1,\omega}(\omega')$, the Kähler-Ricci flow will deform it exponentially fast to a Kähler-Einstein metric in the limit.

Remark 1.2. *The condition that M is pre-stable (cf. Defi 4.14), roughly means that the complex structure doesn't jump in the limit (cf. [6], [23]). In G. Tian's definition of K-stability, this condition appears to be one of three necessary conditions for a complex structure to be K-stable (cf. [13], [28]).*

Remark 1.3. *This gives a sufficient condition for the existence of Kähler-Einstein metrics. More interestingly, by a theorem of G. Tian [28], this also gives a sufficient condition for an algebraic manifold being weakly K-stable. One tempting question is: does this condition imply weak K-stability directly?*

Remark 1.4. *If we call the result in [6] a "pre-baby" step in this ambitious program, then Theorem 1.1 and 1.5 should be viewed as a "baby step" in this program. We wish to remove the assumption on the bound of the bisectional curvature. More importantly (cf. Theorem 1.8 below), we wish to replace the condition on the Ricci curvature in both Theorem 1.1 and 1.5 by a condition on the scalar curvature. Then our theorem really becomes a "small energy" lemma.*

If we remove the condition of "pre-stable", then

Theorem 1.5. *Suppose that $(M, [\omega])$ has no nonzero holomorphic vector fields and E_1 is bounded from below in $[\omega]$. For any $\delta, B, \Lambda > 0$, there exists a small positive constant $\epsilon(\delta, B, \Lambda, \omega) > 0$ such that for any metric ω_0 in the subspace $\mathcal{A}(\delta, B, \Lambda, \epsilon)$ of Kähler metrics*

$$\{\omega_\phi = \omega + \sqrt{-1}\partial\bar\partial\phi \mid Ric(\omega_\phi) > -1 + \delta, \ |\phi| \leq B, \ |Rm|(\omega_\phi) \leq \Lambda, \ E_1(\omega_\phi) \leq \inf E_1 + \epsilon\}$$

the Kähler-Ricci flow will deform it exponentially fast to a Kähler-Einstein metric in the limit.

Remark 1.6. *In light of Theorem 1.7 below, we can replace the condition on E_1 by a corresponding condition on E_0.*

1.3 The relations between energy functionals E_k

Song-Weinkove [26] recently proved that E_k have a lower bound on the space of Kähler metrics with nonnegative Ricci curvature for Kähler-Einstein manifolds. Moreover, they also showed that modulo holomorphic vector fields, E_1 is proper if and only if there exists a Kähler-Einstein metric. Shortly afterwards, N. Pali [20] gave a formula between E_1 and the K-energy E_0, which says that the E_1 energy is always bigger than the K-energy. Tosatti [32] proved that under some curvature assumptions, the critical point of E_k is a Kähler-Einstein metric. Pali's theorem means that E_1 has a lower bound if the K-energy has a lower bound. A natural question is if the converse holds. To our own surprise, we proved the following result.

Theorem 1.7. *E_1 is bounded from below if and only if the K-energy is bounded from below in the class $[\omega]$. Moreover, we have[1]*

$$\inf_{\omega' \in [\omega]} E_{1,\omega}(\omega') = 2 \inf_{\omega' \in [\omega]} E_{0,\omega}(\omega') - \frac{1}{nV} \int_M |\nabla h_\omega|^2 \omega^n,$$

where h_ω is the Ricci potential function with respect to ω.

A crucial observation leads to this theorem is

Theorem 1.8. *Along the Kähler-Ricci flow, E_1 will decrease after finite time.*

Theorem 1.7 and 1.8 of course lead more questions than they answer to. For instance, is the properness of E_k equivalent to the properness of E_l for $k \neq l$? More subtlely, are the corresponding notions of semi-stability ultimately equivalent to each other? Is there a preferred functional in this family, or a better linear combination of these E_k functionals? The first named author

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[1]For simplicity of notation, we will often drop the subscript ϕ and write $|\nabla f|^2$ for $|\nabla f|_\phi^2$. But in an integral, $|\nabla f|^2$ is with respect to the metric of the volume form.

genuinely believes that this observation opens the door for more interesting questions.

Another interesting question is the relation of E_k with various notions of stability defined by algebraic conditions. Theorem 1.1 and 1.5 suggest an indirect link of these functionals E_k and stability. According to A. Futaki [14], these functionals may directly link to the asymptotic Chow semi-stability (note that the right hand side of (1.2) in [14] is precisely equal to $\frac{dE_k}{dt}$ if one takes $p = k + 1$ and $\phi = c_1^k$, cf. Theorem 2.4 below). It is highly interesting to explore further in this direction.

1.4 Main ideas of proofs of Theorem 1.1 and 1.5

In [6], a topological condition is used to control the L^2 norm of the bisectional curvature once the Ricci curvature is controlled. Using the parabolic Moser iteration arguments, this gives a uniform bound on the full bisectional curvature. In the present paper, we need to find a new way to control the full bisectional curvature under the flow. The whole scheme of obtaining this uniform estimate on curvatures depends on two crucial steps and their dynamical interplay.

STEP 1: The first step is to follow the approach of the celebrated work of Yau on the Calabi conjecture (cf. [3], [34]). The key point here is to control the C^0 norm of the evolving Kähler potential $\phi(t)$, in particular, the growth of $u = \frac{\partial\phi}{\partial t}$ along the Kähler-Ricci flow. Note that u satisfies

$$\frac{\partial u}{\partial t} = \triangle_\phi u + u.$$

Therefore, the crucial step is to control the first eigenvalue of the Laplacian operator (assuming the traceless Ricci form is controlled via an iteration process which we will describe as **STEP 2** below). For our purpose, we need to show that the first eigenvalue of the evolving Laplacian operator is bigger than $1 + \gamma$ for some fixed $\gamma > 0$. Such a problem already appeared in [5] and [9] since the first eigenvalue of the Laplacian operator of Kähler-Einstein metrics is exactly 1. If $Aut_r(M, J) \neq 0$, the uniqueness of Kähler-Einstein metrics implies that the dimension of the first eigenspace is fixed; while the vanishing of the Futaki invariant implies that $u(t)$ is essentially perpendicular to the first eigenspace of the evolving metrics. These are two crucial ingredients which allow one to squeeze out a small gap γ on the first eigenvalue estimate. Following the approach in [5] and [9], we can show that u decays exponentially. This in turn implies the C^0 bound on the evolving Kähler potential. Consequently, this leads to control of all derivatives of the evolving potential, in particular, the bisectional curvature. In summary, as long as we have control of the first eigenvalue, one controls the full bisectional curvature of the evolving Kähler metrics.

For Theorem 1.5, a crucial technique step is to use an estimate obtained in [8] on the Ricci curvature tensor.

STEP 2: Here we follow the Moser iteration techniques which appeared in [6]. Assuming that the full bisectional curvature is bounded by some large but fixed number, the norm of the

traceless bisectional curvature and the traceless Ricci tensor both satisfy the following inequality:

$$\frac{\partial u}{\partial t} \leq \triangle_\phi u + |Rm|u \leq \triangle_\phi u + C \cdot u.$$

If the curvature of the evolving metric is controlled in $L^p(p > n)$, then the smallness of the energy E_1 allows you to control the norm of the traceless Ricci tensor of the evolving Kähler metrics (cf. formula (4.2)). According to Theorem 4.8 and 4.16, this will in turn give an improved estimate on the first eigenvalue in a slightly longer period, but perhaps without full uniform control of the bisectional curvature in the "extra" time. However, this gives uniform control on the Kähler potential which in turn gives the bisectional curvature in the extended "extra" time. We use the Moser iteration again to obtain sharper control on the norm of the traceless bisectional curvature.

Hence, the combination of the parabolic Moser iteration together with Yau's estimate, yields the desired global estimate. In comparison, the iteration process in [6] is instructive and more direct. The first named author believes that this approach is perhaps more important than the mild results we obtained there.

1.5   The organization

This paper is roughly organized as follows: In Section 2, we review some basic facts in Kähler geometry and necessary information on the Kähler-Ricci flow. We also include some basic facts on the energy functionals E_k. In Section 3, we prove Theorem 1.7 and 1.8. In Section 4, we prove several technical theorems on the Kähler-Ricci flow. The key results are the estimates of the first eigenvalue of the evolving Laplacian, which are proved in Section 4.3. In Section 5, 6 we prove Theorem 1.1 and Theorem 1.5.

2   Setup and known results

2.1   Setup of notations

Let M be an n-dimensional compact Kähler manifold. A Kähler metric can be given by its Kähler form ω on M. In local coordinates $z_1, \cdots, z_n$, this ω is of the form

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{i\overline{j}} d\, z^i \wedge d\, z^{\overline{j}} > 0,$$

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where $\{g_{i\bar{j}}\}$ is a positive definite Hermitian matrix function. The Kähler condition requires that ω is a closed positive (1,1)-form. In other words, the following holds

$$\frac{\partial g_{i\bar{k}}}{\partial z^j} = \frac{\partial g_{j\bar{k}}}{\partial z^i} \qquad \text{and} \qquad \frac{\partial g_{k\bar{i}}}{\partial z^{\bar{j}}} = \frac{\partial g_{k\bar{j}}}{\partial z^{\bar{i}}} \qquad \forall\, i, j, k = 1, 2, \cdots, n.$$

The Kähler metric corresponding to ω is given by

$$\sqrt{-1}\,\sum_{1}^{n}\, g_{\alpha\bar{\beta}}\, d\,z^{\alpha}\, \otimes d\,z^{\overline{\beta}}.$$

For simplicity, in the following, we will often denote by ω the corresponding Kähler metric. The Kähler class of ω is its cohomology class $[\omega]$ in $H^2(M, \mathbb{R})$. By the Hodge theorem, any other Kähler metric in the same Kähler class is of the form

$$\omega_\phi = \omega + \sqrt{-1}\,\sum_{i,j=1}^{n}\, \frac{\partial^2 \phi}{\partial z^i \partial z^{\bar{j}}}\, dz^i \wedge dz^{\bar{j}} > 0$$

for some real valued function ϕ on M. The functional space in which we are interested (often referred to as the space of Kähler potentials) is

$$\mathcal{P}(M, \omega) = \{\phi \in C^\infty(M, \mathbb{R}) \mid \omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0 \text{ on } M\}.$$

Given a Kähler metric ω, its volume form is

$$\omega^n = n!\,\left(\sqrt{-1}\right)^n \det\left(g_{i\bar{j}}\right) d\,z^1 \wedge d\,z^{\overline{1}} \wedge \cdots \wedge d\,z^n \wedge d\,z^{\overline{n}}.$$

Its Christoffel symbols are given by

$$\Gamma_{i\,j}^k = \sum_{l=1}^{n}\, g^{k\bar{l}}\frac{\partial g_{i\bar{l}}}{\partial z^j} \quad \text{and} \quad \Gamma_{i\,j}^{\bar{k}} = \sum_{l=1}^{n}\, g^{\bar{k}l}\frac{\partial g_{l\bar{i}}}{\partial z^{\bar{j}}}, \qquad \forall\, i, j, k = 1, 2, \cdots n.$$

The curvature tensor is

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial z^{\bar{l}}} + \sum_{p,q=1}^{n}\, g^{p\bar{q}}\frac{\partial g_{i\bar{q}}}{\partial z^k}\frac{\partial g_{p\bar{j}}}{\partial z^{\bar{l}}}, \qquad \forall\, i, j, k, l = 1, 2, \cdots n.$$

We say that ω is of nonnegative bisectional curvature if

$$R_{i\bar{j}k\bar{l}}v^i v^{\bar{j}} w^k w^{\bar{l}} \geq 0$$

for all non-zero vectors v and w in the holomorphic tangent bundle of M. The bisectional curvature and the curvature tensor can be mutually determined. The Ricci curvature of ω is locally given by

$$R_{i\bar{j}} = -\frac{\partial^2 \log \det(g_{k\bar{l}})}{\partial z_i \partial \bar{z}_j}.$$

So its Ricci curvature form is

$$\mathrm{Ric}(\omega) = \sqrt{-1}\sum_{i,j=1}^{n}\, R_{i\bar{j}}\, d\,z^i \wedge d\,z^{\bar{j}} = -\sqrt{-1}\partial\bar{\partial}\log\,\det(g_{k\bar{l}}).$$

It is a real, closed (1,1)-form. Recall that $[\omega]$ is called a canonical Kähler class if this Ricci form is cohomologous to $\lambda\omega$ for some constant λ. In our setting, we require $\lambda = 1$.

2.2 The Kähler-Ricci flow

Let us assume that the first Chern class $c_1(M)$ is positive. Choose an initial Kähler metric ω in $2\pi c_1(M)$. The normalized Kähler-Ricci flow (cf. [16]) on a Kähler manifold M is of the form

$$\frac{\partial g_{i\bar{j}}}{\partial t} = g_{i\bar{j}} - R_{i\bar{j}}, \qquad \forall\, i,\, j = 1, 2, \cdots, n. \tag{2.1}$$

It follows that on the level of Kähler potentials, the Ricci flow becomes

$$\frac{\partial \phi}{\partial t} = \log \frac{\omega_\phi{}^n}{\omega^n} + \phi - h_\omega, \tag{2.2}$$

where h_ω is defined by

$$\mathrm{Ric}(\omega) - \omega = \sqrt{-1}\partial\bar{\partial}h_\omega, \text{ and } \int_M (e^{h_\omega} - 1)\omega^n = 0.$$

Then the evolution equation for bisectional curvature is

$$\begin{aligned}
\frac{\partial}{\partial t}R_{i\bar{j}k\bar{l}} &= \triangle R_{i\bar{j}k\bar{l}} + R_{i\bar{j}p\bar{q}}R_{q\bar{p}k\bar{l}} - R_{i\bar{p}k\bar{q}}R_{p\bar{j}q\bar{l}} + R_{i\bar{l}p\bar{q}}R_{q\bar{p}k\bar{j}} + R_{i\bar{j}k\bar{l}} \\
&\quad - \frac{1}{2}\left(R_{i\bar{p}}R_{p\bar{j}k\bar{l}} + R_{p\bar{j}}R_{i\bar{p}k\bar{l}} + R_{k\bar{p}}R_{i\bar{j}p\bar{l}} + R_{p\bar{l}}R_{i\bar{j}k\bar{p}} \right).
\end{aligned} \tag{2.3}$$

Here Δ is the Laplacian of the metric $g(t)$. The evolution equation for Ricci curvature and scalar curvature are

$$\frac{\partial R_{i\bar{j}}}{\partial t} = \triangle R_{i\bar{j}} + R_{i\bar{j}p\bar{q}}R_{q\bar{p}} - R_{i\bar{p}}R_{p\bar{j}}, \tag{2.4}$$

$$\frac{\partial R}{\partial t} = \triangle R + R_{i\bar{j}}R_{j\bar{i}} - R. \tag{2.5}$$

For direct computations and using the evolving frames, we can obtain the following evolution equations for the bisectional curvature:

$$\frac{\partial R_{i\bar{j}k\bar{l}}}{\partial t} = \Delta R_{i\bar{j}k\bar{l}} - R_{i\bar{j}k\bar{l}} + R_{i\bar{j}m\bar{n}}R_{n\bar{m}k\bar{l}} - R_{i\bar{m}k\bar{n}}R_{m\bar{j}n\bar{l}} + R_{i\bar{l}m\bar{n}}R_{n\bar{m}k\bar{l}} \tag{2.6}$$

As usual, the flow equation (2.1) or (2.2) is referred to as the Kähler-Ricci flow on M. It was proved by Cao [3], who followed Yau's celebrated work [34], that the Kähler-Ricci flow exists globally for any smooth initial Kähler metric. It was proved by S. Bando [1] in dimension 3 and N. Mok [19] in all dimensions that the positivity of the bisectional curvature is preserved under the flow. In [9] and [10], the first named author and G. Tian proved that the Kähler-Ricci flow, in a Kähler-Einstein manifold, initiated from a metric with positive bisectional curvature converges to a Kähler-Einstein metric with constant bisectional curvature. In unpublished work on the Kähler-Ricci flow, G. Perelman proved, along with other results, that the scalar curvature is always uniformly bounded.

2.3 Energy functionals E_k

In [9], a family of energy functionals $E_k(k = 0, 1, 2, \cdots, n)$ was introduced and these functionals played an important role there. First, we recall the definitions of these functionals.

Definition 2.1. *For any $k = 0, 1, \cdots, n$, we define a functional E_k^0 on $\mathcal{P}(M, \omega)$ by*

$$E_{k,\omega}^0(\phi) = \frac{1}{V} \int_M \left(\log \frac{\omega_\phi^n}{\omega^n} - h_\omega \right) \left(\sum_{i=0}^{k} Ric(\omega_\phi)^i \wedge \omega^{k-i} \right) \wedge \omega_\phi^{n-k} + \frac{1}{V} \int_M h_\omega \left(\sum_{i=0}^{k} Ric(\omega)^i \wedge \omega^{k-i} \right) \wedge \omega^{n-k}.$$

Definition 2.2. *For any $k = 0, 1, \cdots, n$, we define $J_{k,\omega}$ as follows*

$$J_{k,\omega}(\phi) = -\frac{n-k}{V} \int_0^1 \int_M \frac{\partial \phi(t)}{\partial t} (\omega_{\phi(t)}^{k+1} - \omega^{k+1}) \wedge \omega_{\phi(t)}^{n-k-1} \wedge dt,$$

where $\phi(t)(t \in [0, 1])$ is a path from 0 to ϕ in $\mathcal{P}(M, \omega)$.

Definition 2.3. *For any $k = 0, 1, \cdots, n$, the functional $E_{k,\omega}$ is defined as follows*

$$E_{k,\omega}(\phi) = E_{k,\omega}^0(\phi) - J_{k,\omega}(\phi).$$

For simplicity, we will often drop the subscript ω.

By direct computation, we have

Theorem 2.4. *For any $k = 0, 1, 2, \cdots, n$, we have*

$$\frac{dE_k}{dt} = \frac{k+1}{V} \int_M \Delta_\phi \dot\phi Ric(\omega_\phi)^k \wedge \omega_\phi^{n-k}$$
$$-\frac{n-k}{V} \int_M \dot\phi (Ric(\omega_\phi)^{k+1} - \omega_\phi^{k+1}) \wedge \omega_\phi^{n-k-1}.$$

Here $\phi(t)$ is any path in $\mathcal{P}(M, \omega)$.

Remark 2.5. *Note that*

$$\frac{dE_0}{dt} = -\frac{n}{V} \int_M \dot\phi (Ric(\omega_\phi) - \omega_\phi) \wedge \omega_\phi^{n-1}.$$

Thus, E_0 is the well-known K-energy.

Theorem 2.6. *Along the Kähler-Ricci flow where $Ric(\omega_\phi) > -\omega_\phi$ is preserved, we have*

$$\frac{dE_k}{dt} \leq -\frac{k+1}{V} \int_M (R(\omega_\phi) - r) Ric(\omega_\phi)^k \wedge \omega_\phi^{n-k}.$$

When $k = 0, 1$, we have

$$\frac{dE_0}{dt} = -\frac{1}{V} \int_M |\nabla \dot\phi|^2 \omega_\phi^n \leq 0,$$
$$\frac{dE_1}{dt} \leq -\frac{2}{V} \int_M (R(\omega_\phi) - r)^2 \omega_\phi^n \leq 0.$$

Recently, Pali in [20] found the following formula, which will be used in this paper.

Theorem 2.7. *For any* $\phi \in \mathcal{P}(M, \omega)$, *we have*

$$E_{1,\omega}(\phi) = 2E_{0,\omega}(\phi) + \frac{1}{nV} \int_M |\nabla u|^2 \omega_\phi^n - \frac{1}{nV} \int_M |\nabla h_\omega|^2 \omega^n,$$

where

$$u = \log \frac{\omega_\phi^n}{\omega^n} + \phi - h_\omega.$$

Remark 2.8. *This formula directly implies that if E_0 is bounded from below, then E_1 is bounded from below on $\mathcal{P}(M, \omega)$.*

Remark 2.9. *In a forthcoming paper [18], the second named author will generalize Theorem 2.7 to all the functionals $E_k(k \geq 1)$, and discuss some interesting relations between E_k.*

3 Energy functionals E_0 and E_1

In this section, we want to prove Theorem 1.7 and 1.8.

3.1 Energy functional E_1 along the Kähler-Ricci flow

The following theorem is well-known in literature (cf. [7]).

Lemma 3.1. *The minimum of the scalar curvature along the Kähler-Ricci flow, if negative, will increase to zero exponentially.*

Proof. Let $\mu(t) = -\min_M R(x, 0)e^{-t}$, then

$$\begin{aligned} \frac{\partial}{\partial t}(R + \mu(t)) &= \Delta(R + \mu(t)) + |Ric|^2 - (R + \mu(t)) \\ &\geq \Delta(R + \mu(t)) - (R + \mu(t)). \end{aligned}$$

Since $R(x, 0) + \mu(0) \geq 0$, by the maximum principle we have

$$R(x, t) \geq -\mu(t) = \min_M R(x, 0)e^{-t}.$$

$\square$

Using the above lemma, the following theorem is an easy corollary of Pali's formula.

Theorem 3.2. *Along the Kähler-Ricci flow, E_1 will decrease after finite time. In particular, if the initial scalar curvature $R(0) > -n + 1$, then there is a small constant $\delta > 0$ depending only on n and $\min_{x \in M} R(0)$ such that for all time $t > 0$, we have*

$$\frac{d}{dt}E_1 \leq -\frac{\delta}{V} \int_M |\nabla \dot{\phi}|^2 \omega_\phi^n \leq 0. \tag{3.1}$$

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Proof. Along the Kähler-Ricci flow, the evolution equation for $|\nabla \dot{\phi}|^2$ is

$$\frac{\partial}{\partial t}|\nabla \dot{\phi}|^2 = \Delta_\phi |\nabla \dot{\phi}|^2 - |\nabla\nabla \dot{\phi}|^2 - |\nabla\bar{\nabla} \dot{\phi}|^2 + |\nabla \dot{\phi}|^2.$$

By Theorem 2.7, we have

$$
\begin{aligned}
\frac{d}{dt}E_1 &= -\frac{2}{V}\int_M |\nabla \dot{\phi}|^2\omega_\phi^n + \frac{1}{nV}\frac{d}{dt}\int_M |\nabla \dot{\phi}|^2\omega_\phi^n \\
&= -\frac{2}{V}\int_M |\nabla \dot{\phi}|^2\omega_\phi^n + \frac{1}{nV}\int_M (-|\nabla\nabla \dot{\phi}|^2 - |\nabla\bar{\nabla} \dot{\phi}|^2 + |\nabla \dot{\phi}|^2 + |\nabla \dot{\phi}|^2(n-R))\omega_\phi^n
\end{aligned}
$$

If the scalar curvature at the initial time $R(x,0) \geq -n + 1 + n\delta(n \geq 2)$ for some small $\delta > 0$, by Lemma 3.1 for all time $t > 0$ we have $R(x,t) \geq -n + 1 + n\delta$. Then we have

$$
\begin{aligned}
\frac{d}{dt}E_1 &\leq -\frac{2}{V}\int_M |\nabla \dot{\phi}|^2\omega_\phi^n + \frac{1}{nV}\int_M (-|\nabla\nabla \dot{\phi}|^2 - |\nabla\bar{\nabla} \dot{\phi}|^2 + |\nabla \dot{\phi}|^2 + |\nabla \dot{\phi}|^2(2n-1-n\delta))\omega_\phi^n \\
&\leq -\frac{\delta}{V}\int_M |\nabla \dot{\phi}|^2\omega_\phi^n.
\end{aligned}
\tag{3.2}
$$

Otherwise, by Lemma 3.1 after finite time $T_0 = \log\frac{-\min_M R(x,0)}{n-1-n\delta}$, we still have $R(x,t) > -n + 1 + n\delta$ for small $\delta > 0$. Thus, the inequality (3.2) holds.

If $n = 1$, by direct calculation we have

$$\frac{dE_1}{dt} = -\frac{2}{V}\int_M (R(\omega_\phi) - 1)^2\omega_\phi = -\frac{2}{V}\int_M (\Delta_\phi\dot{\phi})^2\omega_\phi.$$

If the initial scalar curvature $R(0) > 0$, by R. Hamilton's results in [17] the scalar curvature has a uniformly positive lower bound. Thus, $R(t) \geq c > 0$ for some constant $c > 0$ and all $t > 0$. Therefore, by the proof of Lemma 4.13 in section 4.3.1 the first eigenvalue of Δ_ϕ satisfies $\lambda_1(t) \geq c$. Then

$$\frac{dE_1}{dt} = -\frac{2}{V}\int_M (\Delta_\phi\dot{\phi})^2\omega_\phi \leq -\frac{2c}{V}\int_M |\nabla \dot{\phi}|^2\omega_\phi.$$

The theorem is proved.

$\square$

3.2 On the lower bound of E_0 and E_1

In this section, we will prove Theorem 1.7. Recall the generalized energy:

$$
\begin{aligned}
I_\omega(\phi) &= \frac{1}{V}\sum_{i=0}^{n-1}\int_M \sqrt{-1}\partial\phi \wedge \bar{\partial}\phi \wedge \omega^i \wedge \omega_\phi^{n-1-i}, \\
J_\omega(\phi) &= \frac{1}{V}\sum_{i=0}^{n-1}\frac{i+1}{n+1}\int_M \sqrt{-1}\partial\phi \wedge \bar{\partial}\phi \wedge \omega^i \wedge \omega_\phi^{n-1-i}.
\end{aligned}
$$

By direct calculation, we can prove

$$0 \leq I - J \leq I \leq (n+1)(I-J)$$

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and for any Kähler potential $\phi(t)$

$$\frac{d}{dt}(I_\omega - J_\omega)(\phi(t)) = -\frac{1}{V} \int_M \phi \Delta_\phi \dot\phi \, \omega_\phi^n.$$

The behaviour of E_1 for the family of Kähler potentials $\phi(t)$ satisfying the equation (3.3) below has been studied by Song and Weinkove in [26]. Following their ideas, we have the following lemma.

Lemma 3.3. *For any Kähler metric $\omega_0 \in [\omega]$, there exists a Kähler metric $\omega_0' \in [\omega]$ such that $Ric(\omega_0') > 0$ and*

$$E_0(\omega_0) \geq E_0(\omega_0').$$

Proof. We consider the complex Monge-Ampère equation

$$(\omega_0 + \sqrt{-1}\partial\bar\partial\varphi)^n = e^{th_0+c_t}\omega_0^n, \tag{3.3}$$

where h_0 satisfies the following equation

$$Ric(\omega_0) - \omega_0 = \sqrt{-1}\partial\bar\partial h_0, \qquad \frac{1}{V} \int_M e^{h_0}\omega_0^n = 1$$

and c_t is the constant chosen so that

$$\int_M e^{th_0+c_t}\omega_0^n = V.$$

By Yau's results in [34], there exists a unique $\varphi(t)(t \in [0,1])$ to the equation (3.3) with $\int_M \varphi\omega_0^n = 0$. Then $\varphi(0) = 0$. Note that the equation (3.3) implies

$$Ric(\omega_\varphi) = \omega_\varphi + (1-t)\sqrt{-1}\partial\bar\partial h_0 - \sqrt{-1}\partial\bar\partial\varphi, \tag{3.4}$$

and

$$\Delta_\varphi\dot\varphi = h_0 + c_t'.$$

By the definition of E_0 we have

$$\begin{aligned}
\frac{d}{dt}E_0(\varphi(t)) &= -\frac{1}{V} \int_M \dot\varphi(R(\omega_\varphi) - n)\omega_\varphi^n \\
&= -\frac{1}{V} \int_M \dot\varphi((1-t)\Delta_\varphi h_0 - \Delta_\varphi\varphi)\omega_\varphi^n \\
&= -\frac{1-t}{V} \int_M \Delta_\varphi\dot\varphi h_0 \, \omega_\varphi^n + \frac{1}{V} \int_M \varphi\Delta_\varphi\dot\varphi\omega_\varphi^n \\
&= -\frac{1-t}{V} \int_M (\Delta_\varphi\dot\varphi)^2 \, \omega_\varphi^n - \frac{d}{dt}(I - J)_{\omega_0}(\varphi).
\end{aligned}$$

Integrating the above formula from 0 to 1, we have

$$E_0(\varphi(1)) - E_0(\omega_0) = -\frac{1}{V} \int_0^1 (1-s) \int_M (\Delta_\varphi\dot\varphi)^2 \, \omega_\varphi^n \wedge ds - (I - J)_{\omega_0}(\varphi(1)) \leq 0.$$

By the equation (3.4), we know $Ric(\omega_{\varphi(1)}) > 0$. This proves the lemma. $\qquad\square$

Now we can prove Theorem 1.7.

Theorem 3.4. E_1 *is bounded from below if and only if the K-energy is bounded from below in the class* $[\omega]$. *Moreover, we have*

$$\inf_{\omega' \in [\omega]} E_1(\omega') = 2 \inf_{\omega' \in [\omega]} E_0(\omega') - \frac{1}{nV} \int_M |\nabla h_\omega|^2 \omega^n.$$

Proof. It is sufficient to show that if E_1 is bounded from below, then E_0 is bounded from below. For any Kähler metric ω_0, by Lemma 3.3 there exists a Kähler metric $\omega_0' = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_0$ such that

$$Ric(\omega_0') \geq c > 0, \qquad E_0(\omega_0) \geq E_0(\omega_0'),$$

where c is a constant depending only on ω_0. Let $\varphi(t)$ be the solution to the Kähler-Ricci flow with the initial metric ω_0',

$$\frac{\partial \varphi}{\partial t} = \log \frac{\omega_\varphi^n}{\omega^n} + \varphi - h_\omega, \qquad \varphi(0) = \varphi_0.$$

Then for any $t > s \geq 0$, by Theorem 3.2 we have

$$E_1(t) - E_1(s) \leq 2\delta(E_0(t) - E_0(s)), \tag{3.5}$$

where $E_1(t) = E_1(\omega, \omega_{\varphi(t)})$ and $\delta = \frac{n-1}{2n}$ if $n \geq 2$, or $\delta = c > 0$ if $n = 1$. Here c is a constant obtained in the proof of Theorem 3.2. By Theorem 2.7 we have

$$E_1(t) - 2E_0(s) - \frac{1}{nV} \int_M |\nabla\dot\varphi|^2 \omega_\varphi^n(s) + C_\omega \leq \delta(E_1(t) - \frac{1}{nV} \int_M |\nabla\dot\varphi|^2 \omega_\varphi^n(t) + C_\omega) - 2\delta E_0(s).$$

i.e.

$$E_1(t) - \frac{1}{n(1-\delta)V} \int_M |\nabla\dot\varphi|^2 \omega_\varphi^n(s) + \frac{\delta}{n(1-\delta)V} \int_M |\nabla\dot\varphi|^2 \omega_\varphi^n(t) + C_\omega \leq 2E_0(s), \tag{3.6}$$

where $C_\omega = \frac{1}{nV} \int_M |\nabla h_\omega|^2 \omega^n$. By (3.5) we know E_0 is bounded from below along the Kähler-Ricci flow. Thus there exists a sequence of times t_m such that

$$\int_M |\nabla\dot\varphi|^2 \omega_\varphi^n(t_m) \to 0, \qquad m \to \infty.$$

We choose $t = t_m$ and let $m \to \infty$ in (3.6),

$$\inf E_1 - \frac{1}{n(1-\delta)V} \int_M |\nabla\dot\varphi|^2 \omega_\varphi^n(s) + C_\omega \leq 2E_0(s) \leq 2E_0(\omega_0'),$$

where the last inequality is because E_0 is decreasing along the Kähler-Ricci flow. Thus, we choose $s = t_m$ again and let $m \to \infty$,

$$\inf E_1 + C_\omega \leq 2E_0(\omega_0') \leq 2E_0(\omega_0).$$

Thus, E_0 is bounded from below in $[\omega]$, and

$$\inf E_1 \leq 2 \inf E_0 - C_\omega.$$

On the other hand, for any $\omega' \in [\omega]$ we have

$$E_1(\omega') \geq 2E_0(\omega') - C_\omega.$$

Combining the last two inequalities, we have $\inf E_1 = 2 \inf E_0 - C_\omega$. Thus, the theorem is proved. $\qquad\square$

4 Some technical lemmas

In this section, we will prove some technical lemmas, which will be used in the proof of Theorem 1.1 and 1.5. These lemmas are based on the Kähler-Ricci flow

$$\frac{\partial \phi}{\partial t} = \log \frac{\omega_\phi^n}{\omega^n} + \phi - h_\omega.$$

Most of these results are taken from [6]-[10]. The readers are referred to these papers for the details. Here we will prove some of them for completeness.

4.1 Estimates of the Ricci curvature

The following result shows that we can control the curvature tensor in a short time.

Lemma 4.1. *(cf. [6]) Suppose that for some $\delta > 0$, the curvature of $\omega_0 = \omega + \sqrt{-1}\partial\bar{\partial}\phi(0)$ satisfies the following conditions*

$$\begin{cases} |Rm|(0) & \leq & \Lambda, \\ R_{i\bar{j}}(0) & \geq & -1 + \delta. \end{cases}$$

Then there exists a constant $T(\delta, \Lambda) > 0$, such that for the evolving Kähler metric $\omega_t (0 \leq t \leq 6T)$, we have the following

$$\begin{cases} |Rm|(t) & \leq & 2\Lambda, \\ R_{i\bar{j}}(t) & \geq & -1 + \frac{\delta}{2}. \end{cases} \tag{4.1}$$

Lemma 4.2. *(cf. [6])If $E_1(0) \leq \inf_{\omega' \in [\omega]} E_1(\omega') + \epsilon$, and*

$$Ric(t) + \omega(t) \geq \frac{\delta}{2} > 0, \qquad \forall t \in [0, T],$$

then along the Kähler-Ricci flow we have

$$\frac{1}{V} \int_0^T \int_M |Ric - \omega|^2(t) \omega_\phi^n \wedge dt \leq \frac{\epsilon}{2}. \tag{4.2}$$

Since we have the estimate of the Ricci curvature, the following theorem shows that the Sobolev constant is uniformly bounded if E_1 is small.

Proposition 4.3. *(cf. [6]) Along the Kähler-Ricci flow, if $E_1(0) \leq \inf_{\tilde{\omega} \in [\omega]} E_1(\tilde{\omega}) + \epsilon$, and for any $t \in [0, T]$,*

$$Ric(t) + \omega(t) \geq 0,$$

the diameter of the evolving metric ω_ϕ is uniformly bounded for $t \in [0, T]$. As $\epsilon \to 0$, we have $D \to \pi$. Let $\sigma(\epsilon)$ be the maximum of the Sobolev and Poincaré constants with respect to the metric ω_ϕ. As $\epsilon \to 0$, we have $\sigma(\epsilon) \leq \sigma < +\infty$. Here σ is a constant independent of ϵ.

Next we state a parabolic version of Moser iteration argument (cf. [10]).

Proposition 4.4. *Suppose the Sobolev and Poincaré constants of the evolving Kähler metrics $g(t)$ are both uniformly bounded by σ. If a nonnegative function u satisfies the following inequality*

$$\frac{\partial}{\partial t}u \leq \Delta_\phi u + f(t,x)u, \quad \forall\, t \in (a,b),$$

where $|f|_{L^p(M,g(t))}$ is uniformly bounded by some constant c for some $p > \frac{m}{2}$, where $m = 2n = \dim_\mathbb{R} M$, then for any $\tau \in (0, b-a)$ and any $t \in (a+\tau, b)$, we have[2]

$$u(t) \leq \frac{C(n,\sigma,c)}{\tau^{\frac{m+2}{4}}}\Big(\int_{t-\tau}^t \int_M u^2\,\omega_\phi^n \wedge ds\Big)^{\frac{1}{2}}.$$

By the above Moser iteration, we can show the following lemma.

Lemma 4.5. *For any $\delta, \Lambda > 0$, there exists a small positive constant $\epsilon(\delta, \Lambda) > 0$ such that if the initial metric ω_0 satisfies the following condition:*

$$Ric(0) > -1 + \delta, \ |Rm(0)| \leq \Lambda, \ E_1(0) \leq \inf E_1 + \epsilon, \tag{4.3}$$

then after time $2T$ along the Kähler-Ricci flow, we have

$$|Ric - \omega|(t) \leq C_1(T,\Lambda)\epsilon, \qquad \forall t \in [2T, 6T] \tag{4.4}$$

and

$$|\dot\phi - c(t)|_{C^0} \leq C(\sigma)C_1(T,\Lambda)\epsilon, \qquad \forall t \in [2T, 6T], \tag{4.5}$$

where $c(t)$ is the average of $\dot\phi$ with respect to the metric $g(t)$, and σ is the uniformly upper bound of the Sobolev and Poincaré constants in Proposition 4.3.

Proof. Let $Ric^0 = Ric - \omega$. Then $u = |Ric^0|^2(t)$ satisfies the parabolic inequality

$$\frac{\partial u}{\partial t} \leq \Delta_\phi u + c(n)|Rm|_{g(t)}u,$$

Note that by Lemma 4.1, $|Rm|(t) \leq 2\Lambda$, for $0 \leq t \leq 6T$. Then applying Lemma 4.1 again and Lemma 4.2 for $t \in [2T, 6T]$, we have [3]

$$
\begin{aligned}
|Ric^0|^2(t) &\leq C(\Lambda, T)\Big(\int_0^{6T}\int_M |Ric - \omega|^4(t)\omega_\phi^n \wedge dt\Big)^{\frac{1}{2}} \\
&\leq C(\Lambda, T)(1+\Lambda)\Big(\int_0^{6T}\int_M |Ric - \omega|^2(t)\omega_\phi^n \wedge dt\Big)^{\frac{1}{2}} \\
&\leq C(\Lambda, T)\sqrt{\epsilon}.
\end{aligned}
$$

Thus,

$$|Ric - \omega|(t) \leq C(\Lambda, T)\epsilon^{\frac{1}{4}}. \tag{4.6}$$

Recall that $\Delta_\phi\dot\phi = n - R(\omega_\phi)$, by the above estimate and Proposition 4.3 we have

$$|\dot\phi - c(t)|_{C^0} \leq C(\sigma)C(T,\Lambda)\epsilon^{\frac{1}{4}}, \forall t \in [2T, 6T]. \tag{4.7}$$

For simplicity, we can write $\epsilon^{\frac{1}{4}}$ in the inequalities (4.6) and (4.7) as ϵ, since we can assume $E_1(0) \leq \inf E_1 + \epsilon^4$ in the assumption. The lemma is proved. □

---

[2]The constant C may differ from line to line. The notation $C(A, B, ...)$ means that the constant C depends only on $A, B, ...$

[3]Since the volume V of the Kähler manifold M is fixed for the metrics in the same Kähler class, the constant $C(T, \Lambda)$ below should depend on V, but we don't specify this for simplicity.

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4.2 Estimate of the average of $\frac{\partial \phi}{\partial t}$

In this section, we want to control $c(t) = \frac{1}{V} \int_M \dot{\phi} \omega_\phi^n$. Here we follow the argument in [9]. Notice that the argument essentially needs the lower bound of the K-energy, which can be obtained by Theorem 1.7 in our case. Observe that for any solution $\phi(t)$ of the Kähler-Ricci flow,

$$\frac{\partial \phi}{\partial t} = \log \frac{\omega_\phi^n}{\omega^n} + \phi - h_\omega,$$

the function $\tilde{\phi}(t) = \phi(t) + Ce^t$ also satisfies the above equation for any constant C. Since

$$\frac{\partial \tilde{\phi}}{\partial t}(0) = \frac{\partial \phi}{\partial t}(0) + C,$$

we have $\tilde{c}(0) = c(0) + C$. Thus we can normalize the solution $\phi(t)$ such that the average of $\dot{\phi}(0)$ is any given constant.

The proof of the following lemma will be used in section 5 and 6, so we include a proof here.

Lemma 4.6. *(cf. [9])Suppose that the K-energy is bounded from below along the Kähler-Ricci flow. Then we can normalize the solution $\phi(t)$ so that*

$$c(0) = \frac{1}{V} \int_0^\infty e^{-t} \int_M |\nabla \dot{\phi}|^2 \omega_\phi^n \wedge dt < \infty.$$

Then for all time $t > 0$, we have

$$0 < c(t), \quad \int_0^\infty c(t)dt < E_0(0) - E_0(\infty),$$

where $E_0(\infty) = \lim_{t \to \infty} E_0(t)$.

Proof. A simple calculation yields

$$c'(t) = c(t) - \frac{1}{V} \int_M |\nabla \dot{\phi}|^2 \omega_\phi^n.$$

Define

$$\epsilon(t) = \frac{1}{V} \int_M |\nabla \dot{\phi}|^2 \omega_\phi^n.$$

Since the K-energy has a lower bound along the Kähler-Ricci flow, we have

$$\int_0^\infty \epsilon(t)dt = \frac{1}{V} \int_0^\infty \int_M |\nabla \dot{\phi}|^2 \omega_\phi^n \wedge dt = E_0(0) - E_0(\infty).$$

Now we normalize our initial value of $c(t)$ as

$$\begin{aligned}
c(0) &= \int_0^\infty \epsilon(t)e^{-t}dt \\
&= \frac{1}{V} \int_0^\infty e^{-t} \int_M |\nabla \dot{\phi}|^2 \omega_\phi^n \wedge dt \\
&\leq \frac{1}{V} \int_0^\infty \int_M |\nabla \dot{\phi}|^2 \omega_\phi^n \wedge dt \\
&= E_0(0) - E_0(\infty).
\end{aligned}$$

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From the equation for $c(t)$, we have

$$(e^{-t}c(t))' = -\epsilon(t)e^{-t}.$$

Thus, we have

$$0 < c(t) = \int_t^\infty \epsilon(\tau)e^{-(\tau-t)}d\tau \le E_0(0) - E_0(\infty)$$

and

$$\lim_{t\to\infty} c(t) = \lim_{t\to\infty} \int_t^\infty \epsilon(\tau)e^{-(\tau-t)}d\tau = 0.$$

Since the K-energy is bounded from below, we have

$$\int_0^\infty c(t)dt = \frac{1}{V}\int_0^\infty \int_M |\nabla\dot\phi|^2\omega_\phi^n \wedge dt - c(0) \le E_0(0) - E_0(\infty).$$

$\square$

Lemma 4.7. *Suppose that E_1 is bounded from below on $\mathcal{P}(M,\omega)$. For any solution $\phi(t)$ of the Kähler-Ricci flow with the initial metric ω_0 satisfying*

$$E_1(0) \le \inf E_1 + \epsilon,$$

after normalization for the Kähler potential $\phi(t)$ of the solution, we have

$$0 < c(t), \quad \int_0^\infty c(t)\omega_\phi^n \le \frac{\epsilon}{2}.$$

Proof. By Theorem 1.7, the K-energy is bounded from below, then one can find a sequence of times $t_m \to \infty$ such that

$$\int_M |\nabla\dot\phi|^2\omega_\phi^n\Big|_{t=t_m} \to 0.$$

By Theorem 2.7, we have

$$E_1(t) = 2E_0(t) + \frac{1}{V}\int_M |\nabla\dot\phi|^2\omega_\phi^n - C_\omega.$$

Then

$$
\begin{aligned}
2(E_0(0) - E_0(t_m)) &= E_1(0) - E_1(t_m) - \frac{1}{V}\int_M |\nabla\dot\phi|^2\omega_\phi^n\Big|_{t=0} + \frac{1}{V}\int_M |\nabla\dot\phi|^2\omega_\phi^n\Big|_{t=t_m} \\
&\le \epsilon + \frac{1}{V}\int_M |\nabla\dot\phi|^2\omega_\phi^n\Big|_{t=t_m} \\
&\to \epsilon.
\end{aligned}
$$

Since the K-energy is decreasing along the Kähler-Ricci flow, we have

$$E_0(0) - E_0(\infty) \le \frac{\epsilon}{2}.$$

By the proof of Lemma 4.6, for any solution of the Kähler-Ricci flow we can normalize $\phi(t)$ such that

$$0 < c(t), \quad \int_0^\infty c(t)\omega_\phi^n \le E_0(0) - E_0(\infty) \le \frac{\epsilon}{2}.$$

The lemma is proved. $\square$

4.3 Estimate of the first eigenvalue of the Laplacian operator

4.3.1 Case 1: M has no nonzero holomorphic vector fields

In this subsection, we will estimate the first eigenvalue of the Laplacian when M has no nonzero holomorphic vector fields. In order to show that the norms of ϕ decay exponentially in section 4.5, we need to prove that the first eigenvalue is strictly greater than 1.

Theorem 4.8. *Assume that M has no nonzero holomorphic vector fields. For any $A, B > 0$, there exist $\eta(A, B, \omega) > 0$ such that for any metric $\omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi$, if*

$$-\eta\omega_\phi \leq Ric(\omega_\phi) - \omega_\phi \leq A\omega_\phi \quad and \quad |\phi| \leq B,$$

then the first eigenvalue of the Laplacian Δ_ϕ satisfies

$$\lambda_1 > 1 + \gamma(\eta, B, A, \omega),$$

where $\gamma > 0$ depends only on η, B, A and the background metric ω.

The following lemma is taken from [9].

Lemma 4.9. *(cf. [9]) If the Kähler metric ω_ϕ satisfies*

$$Ric(\omega_\phi) \geq \alpha\omega_\phi, \quad and \quad |\phi| \leq B$$

for two constants α and B, then there exists a uniform constant C depending only on α, B and ω such that

$$\inf_M \log \frac{\omega_\phi^n}{\omega^n}(x) \geq -4C(\alpha, B, \Lambda)e^{2(1+\int_M \log \frac{\omega_\phi^n}{\omega^n}\omega_\phi^n)}.$$

The following crucial lemma is taken from Chen-He [8]. Here we include a proof.

Lemma 4.10. *For any constant $A, B > 0$, if $|Ric(\omega_\phi)| \leq A$ and $|\phi| \leq B$, then there is a constant C depending only on A, B and the background metric ω such that $|\phi|_{C^{3,\beta}(M,\omega)} \leq C(A, B, \omega, \beta)$ for any $\beta \in (0, 1)$. In particular, one can find two constants $C_2(A, B, \omega)$ and $C_3(A, B, \omega)$ such that*

$$C_2(A, B, \omega)\omega \leq \omega_\phi \leq C_3(A, B, \omega)\omega.$$

Proof. We use Yau's estimate on complex Monge-Ampère equation to obtain the $C^{3,\beta}$ norm of $|\phi|$. Let $F = \log \frac{\omega_\phi^n}{\omega^n}$. Then we have

$$\Delta_\omega F = g^{i\bar{j}}\partial_i\partial_{\bar{j}} \log \frac{\omega_\phi^n}{\omega^n} = -g^{i\bar{j}}R_{i\bar{j}}(\phi) + R(\omega),$$

where Δ_ω denotes the Laplacian of ω. On the other hand, we choose normal coordinates at a point such that $g_{i\bar{j}} = \delta_{ij}$ and $g_{i\bar{j}}(\phi) = \lambda_i\delta_{ij}$, then

$$g^{i\bar{j}}R_{i\bar{j}}(\phi) = \sum_i R_{i\bar{i}}(\phi) \leq A\sum_i g_{i\bar{i}}(\phi) = A(n + \Delta_\omega\phi)$$

and

$$g^{i\bar{j}}R_{i\bar{j}}(\phi) \geq -A(n + \Delta_\omega\phi).$$

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Hence, we have

$$\Delta_\omega(F - A\phi) \leq R(\omega) + An \tag{4.8}$$

$$\Delta_\omega(F + A\phi) \geq R(\omega) - An. \tag{4.9}$$

Appling the Green formula, we can bound F from above. In fact,

$$
\begin{aligned}
F + A\phi &\leq \frac{1}{V}\int_M -G(x,y)\Delta_\omega(F + A\phi)(y)\omega^n(y) + \frac{1}{V}\int_M (F + A\phi)\omega^n \\
&\leq \frac{1}{V}\int_M -G(x,y)(R(\omega) - An)\omega^n(y) + \frac{1}{V}\int_M (F + A\phi)\omega^n \\
&\leq C(\Lambda, A, B),
\end{aligned}
$$

where Λ is an upper bound of $|Rm|_\omega$. Notice that in the last inequality we used

$$\frac{1}{V}\int_M F\omega^n \leq \log\left(\frac{1}{V}\int_M e^F\omega^n\right) = 0.$$

Hence, $F \leq C(\Lambda, A, B)$. Consider complex Monge-Ampère equation

$$(\omega + \sqrt{-1}\partial\bar\partial\phi)^n = e^F\omega^n, \tag{4.10}$$

by Yau's estimate we have

$$
\begin{aligned}
\Delta_\phi(e^{-k\phi}(n + \Delta_\omega\phi)) &\geq e^{-k\phi}(\Delta_\omega F - n^2\inf_{i\neq j} R_{i\bar{i}j\bar{j}}(\omega)) \\
&\quad - ke^{-k\phi}n(n + \Delta_\omega\phi) + (k + \inf_{i\neq j} R_{i\bar{i}j\bar{j}}(\omega))e^{-k\phi+\frac{-F}{n-1}}(n + \Delta_\omega\phi)^{1+\frac{1}{n-1}} \\
&\geq e^{-k\phi}(R(\omega) - An - \Delta_\omega\phi - n^2\inf_{i\neq j} R_{i\bar{i}j\bar{j}}(\omega)) \\
&\quad - ke^{-k\phi}n(n + \Delta_\omega\phi) + (k + \inf_{i\neq j} R_{i\bar{i}j\bar{j}}(\omega))e^{-k\phi+\frac{-F}{n-1}}(n + \Delta_\omega\phi)^{1+\frac{1}{n-1}}.
\end{aligned}
$$

The function $e^{-k\phi}(n + \Delta_\omega\phi)$ must achieve its maximum at some point p. At this point,

$$0 \geq -An - \Delta_\omega\phi(p) - kn(n + \Delta_\omega\phi) + (k - \Lambda)e^{\frac{-F(p)}{n-1}}(n + \Delta_\omega\phi)^{1+\frac{1}{n-1}}(p).$$

Notice that we can bound $\sup F$ by $C(\Lambda, A, B)$. Thus, the above inequality implies

$$n + \Delta_\omega\phi \leq C_4(\Lambda, A, B).$$

Since we have an upper bound on F, the lower bound of F can be obtained by Lemma 4.9

$$\inf F \geq -4C(\Lambda, A, B)\exp(2 + 2\int_M F\omega_\phi^n) = C(\Lambda, A, B).$$

On the other hand,

$$\inf F \leq \log\frac{\omega_\phi^n}{\omega^n} = \log\prod_i (1 + \phi_{i\bar{i}}) \leq \log(\prod_i (n + \Delta_\omega\phi)^{n-1}(1 + \phi_{i\bar{i}})).$$

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Hence, $1 + \phi_{i\bar{i}} \geq C_5(\Lambda, A, B) > 0$. Thus,

$$C_5(\Lambda, A, B) \leq n + \Delta_\omega \phi \leq C_4(\Lambda, A, B).$$

By (4.8) and (4.9), we have

$$|\Delta_\omega F| \leq C(A, B, \Lambda).$$

By the elliptic estimate, $F \in W^{2,p}(M, \omega)$ for any $p > 1$. Recall that F satisfies the equation (4.10), we have the Hölder estimate $\phi \in C^{2,\alpha}(M, \omega)$ for some $\alpha \in (0,1)$ (cf. [25],[33]). Let ψ be a local potential of ω such that $\omega = \sqrt{-1}\partial\bar{\partial}\psi$. Differential the equation (4.10), we have

$$\Delta_\phi \frac{\partial}{\partial z^i}(\phi + \psi) - \frac{\partial}{\partial z^i}\log\omega^n = \frac{\partial F}{\partial z^i} \in W^{1,p}(M, \omega).$$

Note that the coefficients of Δ_ϕ is in $C^\alpha(M, \omega)$, by the elliptic estimate $\phi \in W^{4,p}(M, \omega)$. Then by the Sobolev embedding theorem for any $\beta \in (0,1)$,

$$|\phi|_{C^{3,\beta}(M,\omega)} \leq C(A, B, \omega, \beta).$$

The lemma is proved.

$\square$

For convenience, we introduce the following definition.

Definition 4.11. *For any Kähler metric ω, we define*

$$W(\omega) = \inf_f \left\{ \int_M |f_{\alpha\beta}|^2 \omega^n \,\Big|\, f \in W^{2,2}(M, \omega), \int_M f^2\omega^n = 1, \int_M f\omega^n = 0 \right\}.$$

Assume that M has no nonzero holomorphic vector fields, then the following lemma gives a positive lower bound of $W(\omega)$.

Lemma 4.12. *Assume that M has no nonzero holomorphic vector fields. For any constant $A, B > 0$, there exists a positive constant C_6 depending on A, B and the background metric ω, such that for any Kähler metric $\omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi$, if*

$$|Ric(\omega_\phi)| \leq A, \quad and \quad |\phi| \leq B,$$

then

$$W(\omega_\phi) \geq C_6 > 0.$$

Proof. Suppose not, we can find a sequence of metrics $\omega_m = \omega + \sqrt{-1}\partial\bar{\partial}\phi_m$ and functions f_m satisfying

$$|Ric(\omega_m)| \leq A, \qquad |\phi_m| \leq B,$$

and

$$\int_M f_m^2 \omega_m^n = 1, \int_M f_m\omega_m^n = 0, \int_M |f_{m,\alpha\beta}|^2_{g_m}\omega_m^n \to 0.$$

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Note that the Sobolev constants with respect to the metrics ω_m are uniformly bounded. By Lemma 4.10, we can assume that ω_m converges to a Kähler metric ω_∞ in $C^{1,\beta}(M,\omega)$ norm for some $\beta \in (0,1)$. Now define a sequence of vector fields

$$X_m^i = g_m^{i\bar{k}} \frac{\partial f_m}{\partial z^k}, \qquad X_m = X_m^i \frac{\partial}{\partial z^i}. \tag{4.11}$$

By direct calculation, we have

$$|X_m|_{g_m}^2 = |\nabla f_m|_{g_m}^2,$$

and

$$\left| \frac{\partial X_m}{\partial \bar{z}} \right|_{g_m}^2 = \sum_{i,j} \left| \frac{\partial X_m^i}{\partial z^{\bar{j}}} \right|_{g_m}^2 = |f_{m,\alpha\beta}|_{g_m}^2.$$

Then

$$\int_M \left| \frac{\partial X_m}{\partial \bar{z}} \right|_{g_m}^2 \omega_{g_m}^n \to 0. \tag{4.12}$$

Next we claim that there exist two positive constants C_7 and C_8 which depend only on A and the Poincaré constant σ such that

$$0 < C_7(\sigma) \leq \int_M |X_m|_{g_m}^2 \omega_{g_m}^n \leq C_8(A). \tag{4.13}$$

In fact, since the Poincaré constant is uniformly bounded in our case,

$$\int_M |X_m|_{g_m}^2 \omega_{g_m}^n = \int_M |\nabla f_m|_{g_m}^2 \omega_{g_m}^n \geq C(\sigma) \int_M f_m^2 \omega_{g_m}^n = C(\sigma).$$

On the other hand, since the Ricci curvature has a upper bound, we have

$$\begin{aligned}
\int_M |\Delta_m f_m|_{g_m}^2 \omega_{g_m}^n &= \int_M |f_{m,\alpha\beta}|_{g_m}^2 \omega_{g_m}^n + \int_M R_{i\bar{j}} f_{m,\bar{i}} f_{m,j} \omega_{g_m}^n \\
&\leq \int_M |f_{m,\alpha\beta}|_{g_m}^2 \omega_{g_m}^n + A \int_M |\nabla f_m|_{g_m}^2 \omega_{g_m}^n \\
&\leq \int_M |f_{m,\alpha\beta}|_{g_m}^2 \omega_{g_m}^n + \frac{1}{2} \int_M |\Delta_m f_m|_{g_m}^2 \omega_{g_m}^n + \frac{A^2}{2} \int_M f_m^2 \omega_{g_m}^n
\end{aligned}$$

Then

$$\int_M |\Delta_m f_m|_{g_m}^2 \leq 1 + A^2.$$

Therefore,

$$\begin{aligned}
\int_M |X_m|_{g_m}^2 \omega_{g_m}^n &= \int_M |\nabla f_m|_{g_m}^2 \omega_{g_m}^n \\
&\leq \frac{1}{2} \int_M |\Delta_m f_m|_{g_m}^2 + \frac{1}{2} \int_M f_m^2 \omega_{g_m}^n \\
&\leq C(A).
\end{aligned}$$

This proves the claim.

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Now we have

$$\int_M f_m^2 \omega_m^n = 1, \qquad \int_M |\nabla \bar{\nabla} f_m|_{g_m}^2 \omega_m^n \le C(A), \quad \int_M |f_{m,\alpha\beta}|_{g_m}^2 \omega_m^n \to 0,$$

then $f_m \in W^{2,2}(M, \omega_m)$. Note that the metrics ω_m are $C^{1,\beta}$ equivalent to ω_∞, then $f_m \in W^{2,2}(M, \omega_\infty)$, thus we can assume f_m strongly converges to f_∞ in $W^{1,2}(M, \omega_\infty)$. By (4.11) X_m strongly converges to X_∞ in $L^2(M, \omega_\infty)$. Thus, by (4.13),

$$0 < C_7 \le \int_M |X_\infty|^2 \omega_\infty^n \le C_8. \tag{4.14}$$

Next we show that X_∞ is holomorphic. In fact, for any vector valued smooth function $\xi = (\xi^1, \xi^2, \cdots, \xi^n)$,

$$
\begin{aligned}
\left| \int_M \xi \cdot \bar{\partial} X_m \omega_\infty^n \right|^2 &= \left| \int_M \xi^k \frac{\partial X_m}{\partial \bar{z}^k} \omega_\infty^n \right|^2 \\
&\le \int_M |\xi|^2 \omega_\infty^n \int_M \left| \frac{\partial X_m}{\partial \bar{z}} \right|^2 \omega_\infty^n \\
&\le C \int_M |\xi|^2 \omega_\infty^n \int_M \left| \frac{\partial X_m}{\partial \bar{z}} \right|_{g_m}^2 \omega_{g_m}^n \to 0.
\end{aligned}
$$

On the other hand,

$$\int_M \xi \cdot \bar{\partial} X_m \omega_\infty^n = - \int_M \bar{\partial} \xi \cdot X_m \omega_\infty^n \to - \int_M \bar{\partial} \xi \cdot X_\infty \omega_\infty^n.$$

Then X_∞ is a weak holomorphic vector field, thus it must be holomorphic. By (4.14) X_∞ is a nonzero holomorphic vector field, which contradicts the assumption that M has no nonzero holomorphic vector fields. The lemma is proved. $\qquad \square$

Lemma 4.13. *If the Kähler metric ω_g satisfies $Ric(\omega_g) \ge (1-\eta)\omega_g$ where $0 < \eta < \frac{\sqrt{C_6}}{2}$. Here C_6 is the constant obtained in Lemma 4.12. Then the first eigenvalue of Δ_g satisfies $\lambda_1 \ge 1+\gamma$, where $\gamma = \frac{\sqrt{C_6}}{2}$.*

Proof. Let u is any eigenfunction of ω_g with eigenvalue λ_1, so $\Delta_g u = -\lambda_1 u$. Then by direct calculation, we have

$$
\begin{aligned}
\int_M u_{ij} u_{\bar{i}\bar{j}} \, \omega_g^n &= - \int_M u_{ij\bar{j}} u_{\bar{i}} \, \omega_g^n \\
&= - \int_M (u_{j\bar{j}i} + R_{i\bar{k}} u_k) u_{\bar{i}} \, \omega_g^n \\
&= \int_M ((\Delta_g u)^2 - R_{i\bar{j}} u_j u_{\bar{i}}) \, \omega_g^n.
\end{aligned}
$$

This implies

$$
\begin{aligned}
C_6 \int_M u^2 \omega_g^n &\le \int_M ((\Delta_g u)^2 - R_{i\bar{j}} u_j u_{\bar{i}}) \, \omega_g^n \\
&\le \lambda_1^2 \int_M u^2 \omega^n - (1-\eta) \int_M |\nabla u|^2 \omega_g^n \\
&= (\lambda_1^2 - (1-\eta)\lambda_1) \int_M u^2 \omega_g^n.
\end{aligned}
$$

Thus, we have $\lambda_1^2 - (1 - \eta)\lambda_1 - C_6 \geq 0$. Then,

$$\lambda_1 \geq 1 + \frac{\sqrt{C_6}}{2}.$$

$\square$

Proof of Theorem 4.8. The theorem follows directly from the above Lemma 4.12 and 4.13.

$\square$

4.3.2   Case 2: M has nonzero holomorphic vector fields

In this subsection, we will consider the case when M has nonzero holomorphic vector fields. Denote by $Aut(M)^\circ$ the connected component containing the identity of the holomorphic transformation group of M. Let K be a maximal compact subgroup of $Aut(M)^\circ$. Then there is a semidirect decomposition of $Aut(M)^\circ$ (cf. [15]),

$$Aut(M)^\circ = Aut_r(M) \propto R_u,$$

where $Aut_r(M) \subset Aut(M)^\circ$ is a reductive algebraic subgroup and the complexification of K, and R_u is the unipotent radical of $Aut(M)^\circ$. Let $\eta_r(M, J)$ be the Lie algebra of $Aut_r(M, J)$.

Now we introduce the following definition which is a mild modification from [6] and [23].

Definition 4.14. *The complex structure J of M is called pre-stable, if no complex structure of the orbit of diffeomorphism group contains larger (reduced) holomorphic automorphism group (i.e., $Aut_r(M)$).*

Now we recall the following $C^{k,\alpha}$ convergence theorem of a sequence of Kähler metrics, which is well-known in literature (cf. [23], [31]).

Theorem 4.15. *Let M be a compact Kähler manifold. Let $(g(t), J(t))$ be any sequence of metrics $g(t)$ and complex structures $J(t)$ such that $g(t)$ is Kähler with respect to $J(t)$. Suppose the following is true:*

1. *For some integer $k \geq 1$, $|\nabla^l Rm|_{g(t)}$ is uniformly bounded for any integer $l(0 \leq l < k)$;*

2. *The injectivity radii $i(M, g(t))$ are all bounded from below;*

3. *There exist two uniform constant c_1 and c_2 such that $0 < c_1 \leq \text{Vol}(M, g(t)) \leq c_2$.*

Then there exists a subsequence of t_j, and a sequence of diffeomorphism $F_j : M \to M$ such that the pull-back metrics $\tilde{g}(t_j) = F_j^* g(t_j)$ converge in $C^{k,\alpha}(\forall \alpha \in (0,1))$ to a $C^{k,\alpha}$ metric g_∞. The pull-back complex structure tensors $\tilde{J}(t_j) = F_j^* J(t_j)$ converge in $C^{k,\alpha}$ to an integral complex structure tensor $\tilde{J}_\infty$. Furthermore, the metric g_∞ is Kähler with respect to the complex structure $\tilde{J}_\infty$.

Theorem 4.16. *Suppose M is pre-stable. For any $\Lambda_0, \Lambda_1 > 0$, there exists $\eta > 0$ depending only on Λ_0 and Λ_1 such that for any metric $\omega \in 2\pi c_1(M)$, if*

$$|Ric(\omega) - \omega| \leq \eta, \quad |Rm(\omega)| \leq \Lambda_0, \quad |\nabla Rm(\omega)| \leq \Lambda_1, \tag{4.15}$$

then for any smooth function f satisfying

$$\int_M f\omega^n = 0 \quad and \quad Re\left(\int_M X(f)\omega^n\right) = 0, \qquad \forall X \in \eta(M, J),$$

we have

$$\int_M |\nabla f|^2 \omega^n > (1 + \gamma(\eta, \Lambda_0, \Lambda_1)) \int_M |f|^2 \omega^n,$$

where $\gamma > 0$ depends only on η, Λ_0 and Λ_1.

Proof. Suppose not, for any positive numbers $\eta_m \to 0$, there exists a sequence of Kähler metrics $\omega_m \in 2\pi c_1(M)$ such that

$$|Ric(\omega_m) - \omega_m| \leq \eta_m, \quad |Rm(\omega_m)| \leq \Lambda_0, \quad |\nabla_m Rm(\omega_m)| \leq \Lambda_1, \tag{4.16}$$

where Rm_m is with respect to the metric ω_m, and smooth functions f_m satisfying

$$\int_M f_m \omega_m^n = 0, \qquad Re\left(\int_M X(f_m)\omega_m^n\right) = 0, \qquad \forall X \in \eta(M, J),$$

$$\int_M |\nabla_m f_m|^2 \omega_m^n < (1 + \gamma_m) \int_M |f_m|^2 \omega_m^n, \tag{4.17}$$

where $0 < \gamma_m \to 0$. Without loss of generality, we may assume that

$$\int_M f_m^2 \omega_m^n = 1, \qquad \forall m \in \mathbb{N},$$

which means

$$\int_M |\nabla_m f_m|^2 \omega_m^n \leq 1 + \gamma_m < 2.$$

Then, f_m will converge in $W^{1,2}$ if (M, ω_m) converges. However, according to our stated condition, (M, ω_m, J) will converge in $C^{2,\alpha}(\alpha \in (0,1))$ to $(M, \omega_\infty, J_\infty)$. In fact, by (4.16) the diameters of ω_m are uniformly bounded. Note that all the metrics ω_m are in the same Kähler class, the volume is fixed. Then by (4.16) again, the injectivity radii are uniformly bounded from below. Therefore, all the conditions of Theorem 4.15 are satisfied.

Note that the complex structure J_∞ lies in the closure of the orbit of diffeomorphisms, while ω_∞ is a Kähler-Einstein metric in (M, J_∞). By the standard deformation theorem in complex structures, we have

$$\dim Aut_r(M, J) \leq \dim Aut_r(M, J_\infty).$$

By abusing notation, we can write

$$Aut_r(M, J) \subset Aut_r(M, J_\infty).$$

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By our assumption of pre-stable of (M, J), we have the inequality the other way around. Thus, we have

$$\dim Aut_r(M, J) = \dim Aut_r(M, J_\infty), \quad \text{or} \quad Aut_r(M, J) = Aut_r(M, J_\infty).$$

Now, let f_∞ be the $W^{1,2}$ limit of f_m, then we have

$$1 \leq |f_\infty|_{W^{1,2}(M, \omega_\infty)} \leq 3$$

and

$$\int_M f_\infty \omega_\infty^n = 0, \qquad Re\left(\int_M X(f_\infty)\omega_\infty^n\right) = 0, \qquad \forall X \in \eta(M, J).$$

Thus, f_∞ is a non-trivial function. Since ω_∞ is a Kähler-Einstein metric, we have

$$\int_M \theta_X f_\infty \omega_\infty^n = 0,$$

where

$$\mathcal{L}_X \omega_\infty = \sqrt{-1}\partial\bar{\partial}\theta_X.$$

This implies that f_∞ is perpendicular to the first eigenspace[4] of $\triangle_{\omega_\infty}$. In other words, there is a $\delta > 0$ such that

$$\int_M |\nabla f_\infty|^2 \omega_\infty^n > (1 + \delta) \int_M f_\infty^2 \omega_\infty^n > 1 + \delta.$$

However, this contradicts the following fact:

$$\begin{aligned}
\int_M |\nabla f_\infty|^2 \omega_\infty^n &\leq \lim_{m\to\infty} \int_M |\nabla_m f_m|^2 \omega_m^n \\
&\leq \lim_{m\to\infty} (1 + \gamma_m) \int_M f_\infty^2 \omega_\infty^n = 1.
\end{aligned}$$

The lemma is then proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.4 Exponential decay in a short time

In this subsection, we will show that the $W^{1,2}$ norm of $\dot{\phi}$ decays exponentially in a short time. Here we follow the argument in [9] and use the estimate of the first eigenvalue obtained in the previous subsection.

Lemma 4.17. *Suppose for any time $t \in [T_1, T_2]$, we have*

$$|Ric - \omega|(t) \leq C_1\epsilon \quad \text{and} \quad \lambda_1(t) \geq 1 + \gamma > 1.$$

Let

$$\mu_0(t) = \frac{1}{V} \int_M (\dot{\phi} - c(t))^2 \omega_\phi^n.$$

If ϵ is small enough, then there exists a constant $\alpha_0 > 0$ depending only on γ, σ and $C_1\epsilon$ such that

$$\mu_0(t) \leq e^{-\alpha_0(t-T_1)}\mu_0(T_1), \qquad \forall t \in [T_1, T_2].$$

---

[4]Note that $\triangle\theta_X = -\theta_X$ is totally real for $X \in Aut_r(M, J_\infty)$. Moreover, the first eigenspace consists of all such θ_X.

Proof. By direct calculation, we have

$$\frac{d}{dt}\mu_0(t) = \frac{2}{V}\int_M (\dot{\phi} - c(t))(\ddot{\phi} - c(t)')\omega_\phi^n + \frac{1}{V}\int_M (\dot{\phi} - c(t))^2 \Delta_\phi\dot{\phi}\omega_\phi^n$$

$$= -\frac{2}{V}\int_M (1 + \dot{\phi} - c(t))|\nabla(\dot{\phi} - c(t))|^2\omega_\phi^n + \frac{2}{V}\int_M (\dot{\phi} - c(t))^2\omega_\phi^n.$$

By the assumption, we have for $t \in [T_1, T_2]$

$$\frac{d}{dt}\mu_0(t) = -\frac{2}{V}\int_M (1 + \dot{\phi} - c(t))|\nabla\dot{\phi}|^2\omega_\phi^n + \frac{2}{V}\int_M (\dot{\phi} - c(t))^2\omega_\phi^n$$

$$\leq -\frac{2}{V}\int_M (1 - C(\sigma)C_1\epsilon)|\nabla\dot{\phi}|^2\omega_\phi^n + \frac{2}{V}\int_M (\dot{\phi} - c(t))^2\omega_\phi^n$$

$$\leq -\frac{2}{V}\int_M (1 - C(\sigma)C_1\epsilon)(1 + \gamma)(\dot{\phi} - c(t))^2\omega_\phi^n + \frac{2}{V}\int_M (\dot{\phi} - c(t))^2\omega_\phi^n$$

$$= -\alpha_0\mu_0(t).$$

Here

$$\alpha_0 = 2(1 - C(\sigma)C_1\epsilon)(1 + \gamma) - 2 > 0,$$

if we choose ϵ small enough. Thus, we have

$$\mu_0(t) \leq e^{-\alpha_0(t-T_1)}\mu_0(T_1).$$

$\square$

Lemma 4.18. *Suppose for any time $t \in [T_1, T_2]$, we have*

$$|Ric - \omega|(t) \leq C_1\epsilon \quad and \quad \lambda_1(t) \geq 1 + \gamma > 1.$$

Let

$$\mu_1(t) = \frac{1}{V}\int_M |\nabla\dot{\phi}|^2\omega_\phi^n.$$

If ϵ is small enough, then there exists a constant $\alpha_1 > 0$ depending only on γ and $C_1\epsilon$ such that

$$\mu_1(t) \leq e^{-\alpha_1(t-T_1)}\mu_1(T_1), \qquad \forall t \in [T_1, T_2].$$

Proof. Recall that the evolution equation for $|\nabla\dot{\phi}|^2$ is

$$\frac{\partial}{\partial t}|\nabla\dot{\phi}|^2 = \Delta_\phi|\nabla\dot{\phi}|^2 - |\nabla\nabla\dot{\phi}|^2 - |\nabla\bar{\nabla}\dot{\phi}|^2 + |\nabla\dot{\phi}|^2.$$

Then for any time $t \in [T_1, T_2]$,

$$\frac{d}{dt}\mu_1(t) = \frac{1}{V}\int_M \left(-|\nabla\nabla\dot{\phi}|^2 - |\nabla\bar{\nabla}\dot{\phi}|^2 + |\nabla\dot{\phi}|^2 + |\nabla\dot{\phi}|^2\Delta_\phi\dot{\phi}\right)\omega_\phi^n$$

$$\leq \frac{1}{V}\int_M \left(-\gamma|\nabla\dot{\phi}|^2 + (n - R(\omega_\phi))|\nabla\dot{\phi}|^2\right)\omega_\phi^n$$

$$\leq -(\gamma - C_1\epsilon)\mu_1(t).$$

Thus, we have

$$\mu_1(t) \leq e^{-\alpha_1(t-T_1)}\mu_1(T_1)$$

where $\alpha_1 = \gamma - C_1\epsilon > 0$ if we choose ϵ small.

$\square$

4.5 Estimate of the C^0 norm of $\phi(t)$

In this subsection, we derive some estimates on the C^0 norm of $|\phi|$. Recall that in the previous subsection we proved that the $W^{1,2}$ norm of $|\dot\phi - c(t)|$ decays exponentially. Based on this result we will use the parabolic Moser iteration to show that the C^0 norm of $|\dot\phi - c(t)|$ also decays exponentially.

Lemma 4.19. *Suppose that $\mu_0(t), \mu_1(t)$ decay exponentially for $t \in [T_1, T_2]$ as in Lemma 4.17 and 4.18, then we have*

$$\left|\frac{\partial\phi}{\partial t} - c(t)\right|_{C^0} \le \frac{C_9(n,\sigma)}{\tau^{\frac{m}{4}}}\left(\mu_0(t-\tau) + \frac{1}{\alpha_1^2}\mu_1^2(t-\tau)\right)^{\frac{1}{2}}, \quad \forall\, t \in [T_1 + \tau, T_2]$$

where $m = \dim_{\mathbb{R}} M$ and $\tau < T_2 - T_1$.

Proof. Let $u = \frac{\partial\phi}{\partial t} - c(t)$, the evolution equation for u is

$$\frac{\partial u}{\partial t} = \Delta_\phi u + u + \mu_1(t),$$

where $\mu_1(t) = \frac{1}{V}\int_M |\nabla\dot\phi|^2 \omega_\phi^n$. Note that in the proof of Lemma 4.18, we derived

$$\frac{\partial}{\partial t}\mu_1(t) \le -\alpha_1\mu_1(t).$$

Thus, we have

$$\frac{\partial}{\partial t}(u_+ + \frac{1}{\alpha_1}\mu_1) \le \Delta_\phi(u_+ + \frac{1}{\alpha_1}\mu_1) + (u_+ + \frac{1}{\alpha_1}\mu_1).$$

where $u_+ = \max\{u, 0\}$. Since $u_+ + \frac{1}{\alpha_1}\mu_1$ is a nonnegative function, we can use the parabolic Moser iteration,

$$(u_+ + \frac{1}{\alpha_1}\mu_1)(t) \le \frac{C(n,\sigma)}{\tau^{\frac{m+2}{4}}}\left(\int_{t-\tau}^t \int_M (u_+ + \frac{1}{\alpha_1}\mu_1)^2(s)\, \omega_\phi^n \wedge ds\right)^{\frac{1}{2}}.$$

Since μ_0 and μ_1 are decreasing,

$$(u_+ + \frac{1}{\alpha_1}\mu_1)(t)$$
$$\le \frac{C(n,\sigma)}{\tau^{\frac{m+2}{4}}}\left(\int_{t-\tau}^t (\mu_0(s) + \frac{1}{\alpha_1^2}\mu_1^2(s))ds\right)^{\frac{1}{2}}$$
$$\le \frac{C(n,\sigma)}{\tau^{\frac{m}{4}}}\left(\mu_0(t-\tau) + \frac{1}{\alpha_1^2}\mu_1^2(t-\tau)\right)^{\frac{1}{2}}. \tag{4.18}$$

On the other hand, the evolution equation for $-u$ is

$$\frac{\partial}{\partial t}(-u) = \Delta_\phi(-u) + (-u) - \mu_1(t) \le \Delta_\phi(-u) + (-u).$$

Thus,

$$\frac{\partial}{\partial t}(-u)_+ \le \Delta_\phi(-u)_+ + (-u)_+.$$

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By the parabolic Moser iteration, we have

$$
\begin{aligned}
(-u)_+ &\leq \frac{C(n,\sigma)}{\tau^{\frac{m+2}{4}}} \Big(\int_{t-\tau}^{t} \int_M (-u)_+^2 \omega_\phi^n \wedge ds \Big)^{\frac{1}{2}} \\
&\leq \frac{C(n,\sigma)}{\tau^{\frac{m}{4}}} \mu_0 (t-\tau)^{\frac{1}{2}}.
\end{aligned}
\tag{4.19}
$$

Combining the two inequalities (4.18)(4.19), we obtain the estimate

$$
\Big| \frac{\partial \phi}{\partial t} - c(t) \Big|_{C^0} \leq \frac{C(n,\sigma)}{\tau^{\frac{m}{4}}} \Big(\mu_0(t-\tau) + \frac{1}{\alpha_1^2} \mu_1^2(t-\tau) \Big)^{\frac{1}{2}}.
$$

This proved the lemma. $\qquad\qquad\square$

Lemma 4.20. *Under the same assumptions as in Lemma 4.19, we have*

$$
|\phi(t)| \leq |\phi(T_1+\tau)| + \frac{C_{10}(n,\sigma)}{\alpha \tau^{\frac{m}{4}}} (\sqrt{\mu_0(T_1)} + \frac{1}{\alpha_1}\mu_1(T_1)) + \tilde{C}, \qquad \forall\, t \in [T_1+\tau, T_2].
$$

Here $\tilde{C} = E_0(0) - E_0(\infty)$ is a constant in Lemma 4.6.

Proof.

$$
\begin{aligned}
|\phi(t)| &\leq |\phi(T_1+\tau)| + \int_{T_1+\tau}^{t} \Big| \frac{\partial \phi(s)}{\partial s} - c(s) \Big| ds + \int_{T_1+\tau}^{t} c(s) ds \\
&\leq |\phi(T_1+\tau)| + \frac{C(n,\sigma)}{\tau^{\frac{m}{4}}} \int_{T_1+\tau}^{t} \Big(\mu_0(s-\tau) + \frac{1}{\alpha_1^2}\mu_1^2(s-\tau) \Big)^{\frac{1}{2}} ds + \tilde{C} \\
&\leq |\phi(T_1+\tau)| + \frac{C(n,\sigma)}{\tau^{\frac{m}{4}}} (\sqrt{\mu_0(T_1)} + \frac{1}{\alpha_1}\mu_1(T_1)) \int_{T_1+\tau}^{t} e^{-\alpha(s-\tau-T_1)} ds + \tilde{C} \\
&\leq |\phi(T_1)| + \frac{C(n,\sigma)}{\alpha \tau^{\frac{m}{4}}} (\sqrt{\mu_0(T_1)} + \frac{1}{\alpha_1}\mu_1(T_1)) + \tilde{C}
\end{aligned}
$$

where $\alpha = \min\{\frac{\alpha_0}{2}, \alpha_1\}$ and $\tilde{C} = E_0(0) - E_0(\infty)$ is a constant in Lemma 4.6. $\qquad\square$

4.6 Estimate of the C^k norm of $\phi(t)$

In this subsection, we shall obtain uniform C^k bounds for the solution $\phi(t)$ of the Kähler-Ricci flow

$$
\frac{\partial \phi}{\partial t} = \log \frac{\omega_\phi^n}{\omega^n} + \phi - h_\omega
$$

with respect to any background metric ω. For simplicity, we normalize h_ω to satisfy

$$
\int_M h_\omega\, \omega^n = 0.
$$

The following is the main result in this subsection.

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Theorem 4.21. *For any positive constants $\Lambda, B > 0$ and small $\eta > 0$, there exists a constant C_{11} depending only on B, η, Λ and the Sobolev constant σ such that if the background metric ω satisfies*

$$|Rm(\omega)| \leq \Lambda, \qquad |Ric(\omega) - \omega| \leq \eta,$$

and $|\phi(t)|, |\dot{\phi}(t)| \leq B$, then

$$|Rm|(t) \leq C_{11}(B, \Lambda, \eta, \sigma).$$

Proof. Note that $R(\omega) - n = \Delta_\omega h_\omega$, by the assumption we have

$$|\Delta_\omega h_\omega| \leq \eta.$$

Since the Sobolev constant with respect to the metric ω is uniformly bounded by a constant σ, we have

$$|h_\omega|_{C^0} \leq C(\sigma)\eta.$$

Now we use Yau's estimate to obtain higher order estimate of ϕ. Define

$$F = \dot{\phi} - \phi + h_\omega,$$

then the Kähler-Ricci flow can be written as

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^F \omega^n.$$

By Yau's estimate we have

$$
\begin{aligned}
\Delta_\phi(e^{-k\phi}(n + \Delta_\omega\phi)) \;\geq\; & e^{-k\phi}(\Delta_\omega F - n^2 \inf_{i \neq j} R_{i\bar{i}j\bar{j}}(\omega)) \\
& - ke^{-k\phi}n(n + \Delta_\omega\phi) + (k + \inf_{i \neq j} R_{i\bar{i}j\bar{j}}(\omega))e^{-k\phi + \frac{-F}{n-1}}(n + \Delta_\omega\phi)^{1 + \frac{1}{n-1}}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\frac{\partial}{\partial t}(e^{-k\phi}(n + \Delta_\omega\phi)) \;=\; & -k\dot{\phi}e^{-k\phi}(n + \Delta_\omega\phi) + e^{-k\phi}\Delta_\omega\dot{\phi} \\
\;=\; & -k\dot{\phi}e^{-k\phi}(n + \Delta_\omega\phi) + e^{-k\phi}\Delta_\omega(F + \phi - h_\omega).
\end{aligned}
$$

Combing the above two inequalities, we have

$$
\begin{aligned}
(\Delta_\phi - \frac{\partial}{\partial t})(e^{-k\phi}(n + \Delta_\omega\phi)) \;\geq\; & e^{-k\phi}(\Delta_\omega h_\omega + n - n^2 \inf_{i \neq j} R_{i\bar{i}j\bar{j}}(\omega)) \\
& + (k\dot{\phi} - kn - 1)e^{-k\phi}(n + \Delta_\omega\phi) \\
& + (k + \inf_{i \neq j} R_{i\bar{i}j\bar{j}}(\omega))e^{-k\phi + \frac{-F}{n-1}}(n + \Delta_\omega\phi)^{1 + \frac{1}{n-1}}.
\end{aligned}
$$

Since $\phi, \Delta_\omega h_\omega, |h_\omega|, |Rm(\omega)|, \dot{\phi}$ are bounded, by the maximum principle we can obtain the following estimate

$$n + \Delta_\omega\phi \leq C_{12}(B, \eta, \Lambda, \sigma).$$

By the definition of F,

$$\log \frac{\omega_\phi^n}{\omega^n} = F \geq -C_{13}(B, \eta, \sigma).$$

On the other hand, we have

$$\log \frac{\omega_\phi^n}{\omega^n} = \log \prod_{i=1}^n (1 + \phi_{i\bar{i}}) \leq \log((n + \Delta_\omega \phi)^{n-1}(1 + \phi_{i\bar{i}})).$$

Thus, $1 + \phi_{i\bar{i}} \geq e^{-C_{13}} C_{12}^{-\frac{1}{n-1}}$, i.e. $C_{14}\omega \leq \omega_\phi \leq C_{12}\omega$. Following Calabi's computation (cf. [10],[34]), we can obtain the following C^3 estimate:

$$|\phi|_{C^3(M,\omega)} \leq C_{14}(B, \eta, \Lambda, \sigma).$$

Since the metrics ω_ϕ are uniformly equivalent, the flow is uniform parabolic with C^1 coefficients. By the standard parabolic estimates, the C^4 norm of ϕ is bounded, and then all the curvature tensors are also bounded. The theorem is proved. $\square$

5 Proof of Theorem 1.5

In this section, we shall prove Theorem 1.5. This theorem needs the technical condition that M has no nonzero holomorphic vector fields, which will be removed in Section 6. The idea is to use the estimate of the first eigenvalue proved in Section 4.3.1.

Theorem 5.1. *Suppose that M has no nonzero holomorphic vector fields and E_1 is bounded from below in $[\omega]$. For any $\delta, B, \Lambda > 0$, there exists a small positive constant $\epsilon(\delta, B, \Lambda, \omega) > 0$ such that for any metric ω_0 in the subspace $\mathcal{A}(\delta, B, \Lambda, \epsilon)$ of Kähler metrics*

$$\{\omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi \mid Ric(\omega_\phi) > -1 + \delta, \ |\phi| \leq B, \ |Rm|(\omega_\phi) \leq \Lambda, \ E_1(\omega_\phi) \leq \inf E_1 + \epsilon\}$$

the Kähler-Ricci flow will deform it exponentially fast to a Kähler-Einstein metric in the limit.

Proof. Let $\omega_0 = \omega + \sqrt{-1}\partial\bar{\partial}\phi(0) \in \mathcal{A}(\delta, B, \Lambda, \epsilon)$, where ϵ will be determined later. Note that $E_1(0) \leq \inf E_1 + \epsilon$, by Lemma 4.7 we have

$$E_0(0) - E_0(\infty) \leq \frac{\epsilon}{2} < 1.$$

Here we choose $\epsilon < 2$. Therefore, we can normalize the Kähler-Ricci flow such that for the normalized solution $\psi(t)$,

$$0 < c(t), \quad \int_0^\infty c(t)dt < 1, \tag{5.1}$$

where $c(t) = \frac{1}{V} \int_M \dot{\psi}\omega_\psi^n$. Now we give the details on how to normalize the solution to satisfy (5.1). Since $\omega_0 = \omega + \sqrt{-1}\partial\bar{\partial}\phi(0) \in \mathcal{A}(\delta, B, \Lambda, \epsilon)$, by Lemma 4.10 we have

$$C_2(\Lambda, B, \omega)\omega \leq \omega_0 \leq C_3(\Lambda, B, \omega)\omega.$$

By the equation of Kähler-Ricci flow, we have

$$|\dot{\phi}|(0) = \left| \log \frac{\omega_\phi^n}{\omega^n} + \phi - h_\omega \right|_{t=0} \leq C_{16}(\omega, \Lambda, B).$$

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Set $\psi(t) = \phi(t) + C_0 e^t$, where

$$C_0 = \frac{1}{V} \int_0^\infty e^{-t} \int_M |\nabla \dot\phi|^2 \, \omega_\phi^n \wedge dt - \frac{1}{V} \int_M \dot\phi \omega_\phi^n \Big|_{t=0}.$$

Then (5.1) holds and

$$|C_0| \leq 1 + C_{16},$$

and

$$|\psi|(0), \; |\dot\psi|(0) \leq B + 1 + C_{16} := B_0.$$

STEP 1.(Estimates for $t \in [2T_1, 6T_1]$). By Lemma 4.1 there exists a constant $T_1(\delta, \Lambda)$ such that

$$Ric(t) > -1 + \frac{\delta}{2}, \quad and \quad |Rm|(t) \leq 2\Lambda, \qquad \forall t \in [0, 6T_1]. \tag{5.2}$$

By Lemma 4.5 and the equation (5.2), we can choose ϵ small enough so that

$$|Ric - \omega|(t) \leq C_1(T_1, \Lambda)\epsilon < \frac{1}{2}, \qquad \forall t \in [2T_1, 6T_1], \tag{5.3}$$

and

$$|\dot\psi - c(t)| \leq C(\sigma)C_1(T_1, \Lambda)\epsilon < 1, \qquad \forall t \in [2T_1, 6T_1]. \tag{5.4}$$

Then by the inequality (5.1)

$$|\dot\psi|(t) \leq 1 + |c(t)| \leq 2, \qquad \forall t \in [2T_1, 6T_1]. \tag{5.5}$$

Note that the equation for $\dot\psi$ is

$$\frac{\partial}{\partial t}\dot\psi = \Delta_\psi \dot\psi + \dot\psi,$$

we have

$$|\dot\psi|(t) \leq |\dot\psi|(0)e^{2T_1} \leq B_0 e^{2T_1}, \qquad \forall t \in [0, 2T_1]. \tag{5.6}$$

Thus, for any $t \in [2T_1, 6T_1]$ we have

$$\begin{aligned} |\psi|(t) &\leq |\psi|(0) + \int_0^{2T_1} |\dot\psi|ds + \int_{2T_1}^t |\dot\psi|ds \\ &\leq B_0 + 2T_1 B_0 e^{2T_1} + 8T_1, \end{aligned}$$

where the last inequality used (5.5) and (5.6). For simplicity, we define

$$\begin{aligned} B_1 : &= B_0 + 2T_1 B_0 e^{2T_1} + 8T_1 + 2, \\ B_k : &= B_{k-1} + 2, \qquad 2 \leq k \leq 4. \end{aligned}$$

Then

$$|\dot\psi|(t), \; |\psi|(t) \leq B_1, \qquad \forall t \in [2T_1, 6T_1].$$

By Theorem 4.21 we have

$$|Rm|(t) \leq C_{11}(B_1, \Lambda_\omega, 1), \qquad \forall t \in [2T_1, 6T_1],$$

where Λ_ω is an upper bound of curvature tensor with respect to the metric ω, and C_{11} is a constant obtained in Theorem 4.21. Set $\Lambda_0 = C_{11}(B_4, \Lambda_\omega, 1)$, we have

$$|Rm|(t) \leq \Lambda_0, \qquad \forall t \in [2T_1, 6T_1].$$

STEP 2.(Estimate for $t \in [2T_1 + 2T_2, 2T_1 + 6T_2]$). By STEP 1, we have

$$|Ric - \omega|(2T_1) \leq C_1\epsilon < \frac{1}{2} \quad and \quad |Rm|(2T_1) \leq \Lambda_0.$$

By Lemma 4.1, there exists a constant $T_2(\frac{1}{2}, \Lambda_0) \in (0, T_1]$ such that

$$|Rm|(t) \leq 2\Lambda_0, \quad and \quad Ric(t) \geq 0, \qquad \forall t \in [2T_1, 2T_1 + 6T_2].$$

Recall that $E_1 \leq \inf E_1 + \epsilon$, by Lemma 4.2 and Lemma 4.5 there exists a constant $C'_1(T_2, \Lambda_0)$ such that

$$|Ric - \omega|(t) \leq C'_1(T_2, \Lambda_0)\epsilon, \qquad \forall t \in [2T_1 + 2T_2, 2T_1 + 6T_2].$$

Choose ϵ small enough so that $C'_1(T_2, \Lambda_0)\epsilon < \frac{1}{2}$. Then by Lemma 4.5,

$$|\dot{\psi} - c(t)|_{C^0} \leq C(\sigma)C'_1(T_2, \Lambda_0)\epsilon, \qquad \forall t \in [2T_1 + 2T_2, 2T_1 + 6T_2].$$

Choose ϵ small enough so that $C(\sigma)C'_1(T_2, \Lambda_0)\epsilon < 1$. Thus, we can estimate the C^0 norm of ψ for any $t \in [2T_1 + 2T_2, 2T_1 + 6T_2]$

$$\begin{aligned}
|\psi(t)| &\leq |\psi|(2T_1 + 2T_2) + \left|\int_{2T_1 + 2T_2}^t \left(\frac{\partial \psi}{\partial s} - c(s)\right)ds\right| + \left|\int_0^t c(s)ds\right| \\
&\leq B_1 + 4T_2 C(\sigma)C'_1(T_2, \Lambda_0)\epsilon + 1.
\end{aligned}$$

Choose ϵ small enough such that $4T_2 C(\sigma)C'_1(T_2, \Lambda_0)\epsilon < 1$, then

$$|\psi(t)| \leq B_2, \qquad \forall t \in [2T_1 + 2T_2, 2T_1 + 6T_2].$$

Since M has no nonzero holomorphic vector fields, applying Theorem 4.8 for the parameters $\eta = C'_1\epsilon$, $A = 1$, $|\phi| \leq B_4$, if we choose ϵ small enough, there exists a constant $\gamma(C'_1\epsilon, B_4, 1, \omega)$ such that the first eigenvalue of the Laplacian Δ_ψ satisfies

$$\lambda_1(t) \geq 1 + \gamma > 1, \qquad \forall t \in [2T_1 + 2T_2, 2T_1 + 6T_2].$$

STEP 3. In this step, we want to prove the following claim:

Claim 5.2. *For any positive number $S \geq 2T_1 + 6T_2$, if*

$$|Ric - \omega|(t) \leq C'_1(T_2, \Lambda_0)\epsilon < \frac{1}{2} \quad and \quad |\psi(t)| \leq B_3, \qquad \forall t \in [2T_1 + 2T_2, S],$$

then we can extend the solution $g(t)$ to $[2T_1 + 2T_2, S + 4T_2]$ such that the above estimates still hold for $t \in [2T_1 + 2T_2, S + 4T_2]$.

Proof. By the assumption and Lemma 4.5, we have

$$|\dot{\psi}(t) - c(t)|_{C^0} \leq C(\sigma)C_1'(T_2, \Lambda_0)\epsilon, \qquad \forall t \in [2T_1 + 2T_2, S].$$

Note that in step 2, we know that $C(\sigma)C_1'(T_2, \Lambda_0)\epsilon < 1$. Then

$$|\dot{\psi}|(t) \leq 2, \qquad \forall t \in [2T_1 + 2T_2, S].$$

Therefore, $|\psi|, |\dot{\psi}| \leq B_3$. By Theorem 4.21 and the definition of Λ_0, we have

$$|Rm|(t) \leq \Lambda_0, \qquad \forall t \in [2T_1 + 2T_2, S].$$

By Lemma 4.1 and the definition of T_2,

$$|Rm|(t) \leq 2\Lambda_0, \quad Ric(t) \geq 0, \qquad \forall t \in [S - 2T_2, S + 4T_2].$$

Thus, by Lemma 4.2 and Lemma 4.5 we have

$$|Ric - \omega|(t) \leq C_1'(T_2, \Lambda_0)\epsilon, \qquad \forall t \in [S, S + 4T_2],$$

and

$$|\dot{\psi} - c(t)|_{C^0} \leq C(\sigma)C_1'(T_2, \Lambda_0)\epsilon, \qquad \forall t \in [S, S + 4T_2].$$

Then we can estimate the C^0 norm of ψ for $t \in [S, S + 4T_2]$,

$$
\begin{aligned}
|\psi(t)| &\leq |\psi|(S) + \left| \int_S^{S+4T_2} \left(\frac{\partial \psi}{\partial s} - c(s) \right) ds \right| + \left| \int_0^\infty c(s)ds \right| \\
&\leq B_3 + 4T_2 C(\sigma)C_1'(T_2, \Lambda_0)\epsilon + 1 \\
&\leq B_4.
\end{aligned}
$$

Then by Theorem 4.8 and the definition of γ, the first eigenvalue of the Laplacian Δ_ψ

$$\lambda_1(t) \geq 1 + \gamma > 1, \qquad \forall t \in [2T_1 + 2T_2, S + 4T_2].$$

Note that

$$\mu_0(2T_1 + 2T_2) = \frac{1}{V} \int_M (\dot{\psi} - c(t))^2 \omega_\psi^n \leq (C(\sigma)C_1'\epsilon)^2$$

and

$$
\begin{aligned}
\mu_1(2T_1 + 2T_2) &= \frac{1}{V} \int_M |\nabla \dot{\psi}|^2 \omega_\psi^n \\
&= \frac{1}{V} \int_M (\dot{\psi} - c(t))(R(\omega_\psi) - n)\omega_\psi^n \\
&\leq C(\sigma)(C_1'\epsilon)^2.
\end{aligned}
$$

By Lemma 4.20, we can choose ϵ small enough such that

$$
\begin{aligned}
|\psi(t)| &\leq |\psi(2T_1 + 3T_2)| + \frac{C(n, \sigma)}{\alpha T_2^{\frac{m}{4}}} \left(\sqrt{\mu_0(2T_1 + 2T_2)} + \frac{1}{\alpha_1} \mu_1(2T_1 + 2T_2) \right) + 1 \\
&\leq B_2 + \frac{C(n, \sigma)}{\alpha T_2^{\frac{m}{4}}} \left(1 + \frac{1}{\alpha_1} C_1'\epsilon \right) C(\sigma)C_1'\epsilon + 1 \\
&\leq B_3
\end{aligned}
$$

for $t \in [S, S + 4T_2]$. Note that ϵ doesn't depend on S here, so it won't become smaller as $S \to \infty$. $\qquad\square$

STEP 4. By step 3, we know the bisectional curvature is uniformly bounded and the first eigenvalue $\lambda_1(t) \geq 1 + \eta > 1$ uniformly for some positive constant $\eta > 0$. Thus, following the argument in [9], the Kähler-Ricci flow converges to a Kähler-Einstein metric exponentially fast. This theorem is proved.

$\square$

6 Proof of Theorem 1.1

In this section, we shall use the pre-stable condition to drop the assumptions that M has no nonzero holomorphic vector fields, and the dependence of the initial Kähler potential. The proof here is roughly the same as in the previous section, but there are some differences.

In the STEP 1 of the proof below, we will choose a new background metric at time $t = 2T_1$, so the new Kähler potential with respect to the new background metric at $t = 2T_1$ is 0, and has nice estimates afterwards. Notice that all the estimates, particularly in Theorem 4.16 and 4.21, are essentially independent of the choice of the background metric. Therefore the choice of ϵ will not depend on the initial Kähler potential $\phi(0)$. This is why we can remove the assumption on the initial Kähler potential.

As in Theorem 1.5, the key point of the proof is to use the improved estimate on the first eigenvalue in Section 4.3.2 (see Claim 6.2 below). Since the curvature tensors are bounded in some time interval, by Shi's estimates the gradient of curvature tensors are also bounded. Then the assumptions of Theorem 4.21 are satisfied, and we can use the estimate of the first eigenvalue.

Now we state the main result of this section.

Theorem 6.1. *Suppose M is pre-stable, and E_1 is bounded from below in $[\omega]$. For any $\delta, \Lambda > 0$, there exists a small positive constant $\epsilon(\delta, \Lambda) > 0$ such that for any metric ω_0 in the subspace $\mathcal{A}(\delta, \Lambda, \omega, \epsilon)$ of Kähler metrics*

$$\{\omega_\phi = \omega + \sqrt{-1}\partial\bar\partial\phi \mid Ric(\omega_\phi) > -1 + \delta, \ |Rm|(\omega_\phi) \leq \Lambda, \ E_1(\omega_\phi) \leq \inf E_1 + \epsilon\},$$

the Kähler-Ricci flow will deform it exponentially fast to a Kähler-Einstein metric in the limit.

Proof. Let $\omega_0 \in \mathcal{A}(\delta, \Lambda, \omega, \epsilon)$, where ϵ will be determined later. By Lemma 4.1 there exists a constant $T_1(\delta, \Lambda)$ such that

$$Ric(t) > -1 + \frac{\delta}{2} \quad and \quad |Rm|(t) \leq 2\Lambda, \qquad \forall t \in [0, 6T_1].$$

By Lemma 4.5, we can choose ϵ small enough so that

$$|Ric - \omega|(t) \leq C_1(T_1, \Lambda)\epsilon < \frac{1}{2}, \qquad \forall t \in [2T_1, 6T_1], \tag{6.1}$$

and

$$|\dot\phi - c(t)| \leq C(\sigma)C_1(T_1, \Lambda)\epsilon < 1, \qquad \forall t \in [2T_1, 6T_1]. \tag{6.2}$$

STEP 1.(Choose a new background metric). Let $\underline{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\phi(2T_1)$ and let $\underline{\phi}(t)$ be the solution to the following Kähler-Ricci flow

$$\begin{cases} \frac{\partial\underline{\phi}(t)}{\partial t} = \log \frac{(\underline{\omega}+\sqrt{-1}\partial\bar{\partial}\underline{\phi})^n}{\underline{\omega}^n} + \underline{\phi} - h_{\underline{\omega}}, & t \geq 2T_1, \\ \underline{\phi}(2T_1) = 0. \end{cases}$$

Here $h_{\underline{\omega}}$ satisfies the following conditions

$$Ric(\underline{\omega}) - \underline{\omega} = \sqrt{-1}\partial\bar{\partial}h_{\underline{\omega}} \quad and \quad \int_M h_{\underline{\omega}}\underline{\omega}^n = 0.$$

Then the metric $\underline{\omega}(t) = \underline{\omega} + \sqrt{-1}\partial\bar{\partial}\underline{\phi}(t)$ satisfies

$$\frac{\partial}{\partial t}\underline{\omega}(t) = -Ric(\underline{\omega}(t)) + \underline{\omega}(t) \quad and \quad \underline{\omega}(2T_1) = \omega + \sqrt{-1}\partial\bar{\partial}\phi(2T_1).$$

By the uniqueness of Kähler-Ricci flow, we have

$$\underline{\omega}(t) = \omega + \sqrt{-1}\partial\bar{\partial}\phi(t), \qquad \forall t \geq 2T_1.$$

Since the Sobolev constant is bounded and

$$|\Delta_{\underline{\omega}}h_{\underline{\omega}}| = |R(\underline{\omega}) - n| \leq C_1(T_1, \Lambda)\epsilon,$$

we have

$$\left|\frac{\partial\underline{\phi}}{\partial t}\right|(2T_1) = |h_{\underline{\omega}}| \leq C(\sigma)C_1(T_1, \Lambda)\epsilon.$$

Since E_1 is decreasing in our case, we have

$$E_1(\underline{\omega}) \leq E_1(\omega + \sqrt{-1}\partial\bar{\partial}\phi(0)) \leq \inf E_1 + \epsilon.$$

By Lemma 4.7, we have

$$E_0(\underline{\omega}) \leq \inf E_0 + \frac{\epsilon}{2}.$$

Thus, by Lemma 4.6 we have

$$\frac{1}{V}\int_{2T_1}^{\infty} e^{-t}\int_M \left|\nabla\frac{\partial\underline{\phi}}{\partial t}\right|^2 \underline{\omega}(t)^n \wedge dt < \frac{\epsilon}{2} < 1.$$

Here we choose $\epsilon < 2$. Set $\psi(t) = \underline{\phi}(t) + C_0 e^{t-2T_1}$, where

$$C_0 = \frac{1}{V}\int_{2T_1}^{\infty} e^{-t}\int_M \left|\nabla\frac{\partial\underline{\phi}}{\partial t}\right|^2 \underline{\omega}(t)^n \wedge dt - \frac{1}{V}\int_M \frac{\partial\underline{\phi}}{\partial t}\underline{\omega}(t)^n\Big|_{t=2T_1}.$$

Then

$$|\psi(2T_1)|, \quad \left|\frac{\partial\psi}{\partial t}\right|(2T_1) \leq 2,$$

and

$$0 < \underline{c}(t), \quad \int_{2T_1}^{\infty} \underline{c}(t)dt < 1,$$

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where $\underline{c}(t) = \frac{1}{V} \int_M \frac{\partial \psi}{\partial t} \omega_\psi^n$. Since

$$|\dot{\psi} - \underline{c}(t)| = |\dot{\phi} - c(t)| \le C(\sigma) C_1(T_1, \Lambda) \epsilon, \qquad \forall t \in [2T_1, 6T_1],$$

we have

$$|\dot{\psi}|(t) \le 2, \qquad \forall t \in [2T_1, 6T_1],$$

and

$$
\begin{aligned}
|\psi|(t) &\le |\psi|(2T_1) + \left| \int_{2T_1}^t (\dot{\psi} - \underline{c}(t)) \right| + \left| \int_{2T}^\infty \underline{c}(s) ds \right| \\
&\le 3 + 4T_1 C(\sigma) C_1(T_1, \Lambda) \epsilon, \qquad \forall t \in [2T_1, 6T_1].
\end{aligned}
$$

Choose ϵ small enough such that $4T_1 C(\sigma) C_1(T_1, \Lambda) \epsilon < 1$, and define $B_k = 2k + 2$. Then

$$|\psi|, |\dot{\psi}| \le B_1, \qquad \forall t \in [2T_1, 6T_1].$$

By Theorem 4.21, we have

$$|Rm|(t) \le C_{11}(B_1, 2\Lambda, 1), \qquad \forall t \in [2T_1, 6T_1].$$

Here C_{11} is a constant obtained in Theorem 4.21. Let $\Lambda_0 := C_{11}(B_3, 2\Lambda, 1)$, then

$$|Rm|(t) \le \Lambda_0, \qquad \forall t \in [2T_1, 6T_1].$$

STEP 2. (Estimates for $t \in [2T_1 + 2T_2, 2T_1 + 6T_2]$). By step 1, we have

$$|Ric - \omega|(2T_1) \le C_1(T_1, \Lambda) \epsilon < \frac{1}{2}, \quad and \quad |Rm|(2T_1) \le \Lambda_0.$$

By Lemma 4.1, there exists a constant $T_2(\frac{1}{2}, \Lambda_0) \in (0, T_1]$ such that

$$|Rm|(t) \le 2\Lambda_0, \quad and \quad Ric(t) \ge 0, \qquad \forall t \in [2T_1, 2T_1 + 6T_2]. \tag{6.3}$$

Recall that $E_1 \le \inf E_1 + \epsilon$, by Lemma 4.2 and Lemma 4.5 there exists a constant $C_1'(T_2, \Lambda_0)$ such that

$$|Ric - \omega|(t) \le C_1'(T_2, \Lambda_0) \epsilon, \qquad \forall t \in [2T_1 + 2T_2, 2T_1 + 6T_2].$$

Choose ϵ small enough so that $C_1'(T_2, \Lambda_0) \epsilon < \frac{1}{2}$. Then by Lemma 4.5,

$$|\dot{\psi}(t) - \underline{c}(t)|_{C^0} \le C(\sigma) C_1'(T_2, \Lambda_0) \epsilon, \qquad \forall t \in [2T_1 + 2T_2, 2T_1 + 6T_2].$$

Choose ϵ small such that $C(\sigma) C_1'(T_2, \Lambda_0) \epsilon < 1$. Thus, we can estimate the C^0 norm of ψ for any $t \in [2T_1 + 2T_2, 2T_1 + 6T_2]$

$$
\begin{aligned}
|\psi(t)| &\le |\psi|(2T_1 + 2T_2) + \left| \int_{2T_1 + 2T_2}^t \left(\frac{\partial \psi}{\partial s} - \underline{c}(s) \right) ds \right| + \left| \int_{2T_1 + 2T_2}^\infty \underline{c}(s) ds \right| \\
&\le B_1 + 4T_2 C(\sigma) C_1'(T_2, \Lambda_0) \epsilon + 1 \\
&\le B_2.
\end{aligned}
$$

Here we choose ϵ small enough such that $4T_2 C(\sigma) C_1'(T_2, \Lambda_0) \epsilon < 1$. Thus, by the definition of Λ_0, we have

$$|Rm|(t) \le \Lambda_0, \qquad \forall t \in [2T_1 + 2T_2, 2T_1 + 6T_2].$$

STEP 3. In this step, we want to prove the following claim:

Claim 6.2. *For any positive number $S \geq 2T_1 + 6T_2$, if*

$$|Ric - \omega|(t) \leq C_1'(T_2, \Lambda_0)\epsilon < \frac{1}{2}, \quad and \quad |Rm|(t) \leq \Lambda_0, \qquad \forall t \in [2T_1 + 2T_2, S],$$

then we can extend the solution $g(t)$ to $[2T_1 + 2T_2, S + 4T_2]$ such that the above estimates still hold for $t \in [2T_1 + 2T_2, S + 4T_2]$.

Proof. By Lemma 4.1 and the definition of T_2,

$$|Rm|(t) \leq 2\Lambda_0, \ Ric(t) \geq 0, \qquad \forall t \in [2T_1 + 2T_2, S + 4T_2].$$

Thus, by Lemma 4.2 and Lemma 4.5 we have

$$|Ric - \omega|(t) \leq C_1'(T_2, \Lambda_0)\epsilon, \qquad \forall t \in [S - 2T_2, S + 4T_2].$$

Therefore, we have

$$|\dot{\psi}(t) - \underline{c}(t)|_{C^0} \leq C(\sigma)C_1'(T_2, \Lambda_0)\epsilon, \qquad \forall t \in [2T_1 + 2T_2, S + 4T_2].$$

By Theorem 1.7 the K-energy is bounded from below, then the Futaki invariant vanishes. Therefore, we have

$$\int_M X(\dot{\psi})\underline{\omega}_\psi^n = 0, \qquad \forall X \in \eta_r(M, J).$$

By the assumption that M is pre-stable and Theorem 4.16, if ϵ is small enough, there exists a constant $\gamma(C_1'\epsilon, 2\Lambda_0)$ such that

$$\int_M |\nabla \dot{\psi}|\underline{\omega}_\psi^n \geq (1 + \gamma) \int_M |\dot{\psi} - \underline{c}(t)|^2 \underline{\omega}_\psi^n.$$

Therefore, Lemma 4.17 still holds, i.e. there exists a constant $\alpha(\gamma, C_1'\epsilon, \sigma) > 0$ such that for any $t \in [2T_1 + 2T_2, S + 4T_2]$

$$\mu_1(t) \leq \mu_1(2T_1 + 2T_2)e^{-\alpha(t - 2T_1 - 2T_2)},$$

and

$$\mu_0(t) \leq \frac{1}{1 - C_1'\epsilon}\mu_1(t) \leq 2\mu_1(2T_1 + 2T_2)e^{-\alpha(t - 2T_1 - 2T_2)}.$$

Then by Lemma 4.20, we can choose ϵ small enough such that

$$\begin{aligned}
|\psi(t)| &\leq |\psi(2T_1 + 3T_2)| + \frac{C_{10}(n, \sigma)}{\alpha T_2^{\frac{m}{4}}}(\sqrt{\mu_0(2T_1 + 2T_2)} + \frac{1}{\alpha_1}\mu_1(2T_1 + 2T_2)) + 1 \\
&\leq B_2 + \frac{C_{10}(n, \sigma)}{\alpha T_2^{\frac{m}{4}}}(1 + \frac{1}{\alpha_1}C_1'\epsilon)C(\sigma)C_1'\epsilon + 1 \\
&\leq B_3
\end{aligned}$$

for $t \in [S, S + 4T_2]$. By the definition of Λ_0, we have

$$|Rm|(t) \leq \Lambda_0, \qquad \forall t \in [S, S + 4T_2].$$

$\square$

STEP 4. By step 3, we know the bisectional curvature is uniformly bounded and the $W^{1,2}$ norm of $\dot{\underline{\phi}} - \underline{c}(t)$ decays exponentially. Thus, following the argument in [9], the Kähler-Ricci flow converges to a Kähler-Einstein metric exponentially fast. This theorem is proved.

$\square$

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